FROM BIG BANG TO BLACK HOLES

The Inflationary Universe, BraneWorld Cosmology and Quantum Gravity

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6 The Inflationary Universe, BraneWorld Cosmology and Quantum Gravity

Stanford physics Professor Andrei Linde, whose theories on the origin of the universe have revolutionized the field of cosmology, has been named co-recipient of the 2002 Dirac Medal by the Abdus Salam International Center for Theoretical Physics (ICTP) in Trieste, Italy. Linde will share the prize with physicists Alan Guth of the Massachusetts Institute of Technology and Paul Steinhardt of Princeton University. In announcing the award, ICTP officials credited the three scientists with developing the concept of inflationary cosmology – the idea that the universe began not with a fiery big bang but with an extraordinarily rapid expansion (inflation) of space in a vacuum-like state.

6.1 Inflation as a New Element of Cosmology

In 1980, an unknown physicist named Alan Guth proposed a modification to the Big Bang theory. Guth suggested that in the first moments of the life of our universe it inflated like an enormous bubble (Fig. 6.2). Inflationary theory has been very successful at solving many of the problems that had puzzled scientists for years, including the fact that the Big Bang would not produce a universe large enough to hold even a sheet of paper.

The reason why something like inflation was needed in cosmology was highlighted by discussions of two key problems in the 1970s. The first of these is the horizon (or causality) problem – the puzzle that the Universe looks the same on opposite sides of the sky (opposite horizons) even though there has not been time since the Big Bang for light (or anything else) to travel across the Universe and back. So how do the opposite horizons “know” how to keep in step with each other? The second puzzle is called the flatness problem. This is the puzzle that the spacetime of the Universe is very nearly flat, which means that the Universe sits just on the dividing line between eternal expansion and eventual recollapse.

In 1981, Alan Guth, then at MIT, published a different version of the inflationary scenario [7], not knowing anything of Starobinsky’s work [23]. This version was more accessible in both senses of the word – it was easier to understand, and Guth was based in the US, able to discuss his ideas freely with colleagues around the world. And as a bonus, Guth came up with the catchy name “inflation” for the process he was describing. There were obvious flaws with the specific details of Guth’s original model (which he acknowledged at the time). In particular, Guth’s model left the Universe after inflation filled with a mess of bubbles (Fig. 6.1), all expanding in their own way and colliding with one another. We see no evidence for these bubbles in the real Universe, so obviously the simplest model of inflation couldn’t be right. But it was this version of the idea that made every cosmologist aware of the power of inflation.

In October 1981, there was an international meeting in Moscow, where inflation was a
Figure 6.1: In this model of inflation the inflaton finds itself trapped in a false minimum. It is freed from this minimum when tunneling is allowed to occur resulting in a first order phase transition in the early universe.

Figure 6.2: Inflationary expansion of the Universe. A closed Universe with initial radius $R_0 \approx \text{Planck scale}$ expands rapidly at around $10^{-43}$ secs and determines then the overall scale of the present Universe. Without inflation, the scale of the Universe has no relation to the Planck scale.
6.1 Inflation as a New Element of Cosmology

major talking point. Stephen Hawking presented a paper claiming that inflation could not be made to work at all, but the Russian cosmologist Andrei Linde presented an improved version, called ”new inflation”, which got around the difficulties with Guth’s model. Ironically, Linde was the official translator for Hawking’s talk, and had the embarrassing task of offering the audience the counter-argument to his own work! But after the formal presentations Hawking was persuaded that Linde was right, and inflation might be made to work after all. Within a few months, the new inflationary scenario was also published by Andreas Albrecht and Paul Steinhardt, of the University of Pennsylvania, and by the end of 1982 inflation was well established. Linde has been involved in most of the significant developments with the theory since then. The next step forward came with the realization that there need not be anything special about the Planck- sized region of spacetime that expanded to become our Universe. If that was part of some larger region of spacetime in which all kinds of scalar fields were at work, then only the regions in which those fields produced inflation could lead to the emergence of a large universe like our own. Linde called this chaotic inflation, because the scalar fields can have any value at different places in the early super-universe; it is the standard version of inflation today, and can be regarded as an example of the kind of reasoning associated with the anthropic principle (but note that this use of the term ”chaos” is like the everyday meaning implying a complicated mess, and has nothing to do with the mathematical subject known as chaos theory).

Figure 6.3: The diagram shows our horizon superimposed on a very large radius sphere on top, or a smaller sphere on the bottom. Since we can only see as far as our horizon, for the inflationary case on top the large radius sphere looks almost flat to us.

The inflationary scenario proposes that the vacuum energy was very large during a brief period early in the history of the Universe. When the Universe is dominated by a vacuum energy density the scale factor grows exponentially, \( a(t) = \exp(H(t - t_i)) \) (Fig. 6.2). The Hubble paraameter really is constant during this epoch. If the inflationary epoch lasts long enough, the exponential function gets very large. This makes \( a(t) \) very large, and thus makes
the radius of curvature of the Universe very large. The diagram of Fig. 6.3 shows our horizon superimposed on a very large radius sphere on top, or a smaller sphere on the bottom. Since we can only see as far as our horizon, for the inflationary case on top the large radius sphere looks almost flat to us.

Very soon after the introduction of inflation by Guth, it was realized that inflation had another remarkable property: it could explain the generation of the primordial density fluctuations in the universe (Fig. 6.4). In inflation, the expansion is so rapid that pairs of virtual particles get ”swept up” in the spacetime and are inflated to causally disconnected regions.

6.2 Models of Inflation

In particle physics, scalar fields are essential elements for the discussion of symmetry breaking processes. The most famous example is the Higgs field in the standard model of particle
6.2 Models of Inflation

Figure 6.5: This space-time diagram shows the inflationary epoch tinted green, and the future lightcones of two events in red. The early event has a future lightcone that covers a huge area, that can easily encompass all of our horizon. Thus we can explain why the temperature of the microwave background is so uniform across the sky.

physics which breaks down the electroweak phase (it provides masses to the $W$ and $Z$ bosons of the $SU(2)_L$ gauge field). Similar fields are postulated in GUTs and SUSY theories.

Inflation can be summarized as follows. Suppose there is a real scalar field $\Phi$ (called the inflaton field) with potential energy $V(\Phi)$ which is quite flat near $\Phi = 0$ and has minima at $\Phi = \Phi_0$ with $V(\Phi_0) = 0$ (Fig. 6.6). At high enough temperatures, $\Phi = 0$ in the Universe due to temperature corrections to $V(\Phi)$. As the temperature drops, the effective potential approaches the $T=0$ potential, but a little potential barrier separating the local minimum at $\Phi = 0$ and the vacua at $\Phi = \Phi_0$ still remains. At some point, $\Phi$ tunnels out to $\Phi_1 \ll \Phi_0$, and a bubble is created with $\Phi = \Phi_1$ in the Universe. The field then rolls over to the minimum very slowly, due to the flatness of the potential. The Lagrangian density

$$L_\Phi = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - V(\Phi)$$

(6.1)

gives the energy–momentum tensor as follows

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial}{\partial g^{\mu\nu}} (\sqrt{-g} L_\Phi).$$

(6.2)

This is equal to

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - g_{\mu\nu} \left( \frac{1}{2} \partial_\rho \Phi \partial^\rho \Phi - V(\Phi) \right).$$

(6.3)

The corresponding density and pressure are given by

$$\rho_\Phi = T_0^0 = \frac{1}{2} \dot{\Phi}^2 + \frac{1}{2} R^{-2}(t)(\nabla \Phi)^2 + V(\Phi)$$

(6.4)

$$P_\Phi = \frac{1}{3} T^i_i = \frac{1}{2} \dot{\Phi}^2 - \frac{1}{6} R^{-2}(t)(\nabla \Phi)^2 - V(\Phi).$$

(6.5)
In order to have an inflationary phase, we have to satisfy the conditions

\[
\dot{\Phi}^2 \ll V(\Phi) \quad (6.6)
\]
\[
(\nabla \Phi)^2 \ll V(\Phi) , \quad (6.7)
\]

so that \( P_\phi = -\rho_\Phi \). Various models have been invented to satisfy these conditions (old inflation, new inflation, chaotic inflation etc). In old inflation, inflation is driven by a first order phase–transition (bubble formation), while new and chaotic inflation are based on second order phase transitions.

**Effective Potential at Finite Temperature**

Since the inflaton field is in thermal equilibrium with a thermal bath, the effective potential will also depend on temperature

\[
V(\Phi, T) = V_0(\Phi) + V_{\text{int}}(\Phi, T) \quad (6.8)
\]

such that for

\[
T \geq T_c : \text{minimum is at } |\Phi| = 0 \quad (6.9)
\]
\[
T \leq T_c : \text{minimum is at } |\Phi| = \sigma , \quad (6.10)
\]

when \( T_C \) denotes the critical temperature of the phase transition (order parameter). A typical form for \( V_0 \) is

\[
V_0(\Phi) = \frac{1}{4} \lambda (\Phi^2 - \sigma^2)^2 . \quad (6.11)
\]

In lowest order, the temperature correction is

\[
V_{\text{int}}(\Phi, T) = \frac{1}{2} g^2 \Phi^2 T^2 . \quad (6.12)
\]

For this potential, the critical temperature is given by \( T_c = \sqrt{\lambda} \sigma / g \), and for \( T \gg T_c \) the field stays in the minimum \( \Phi = 0 \) (a potential of the form (a) in Fig. 6.6). For \( T < T_c \), temperature fluctuations drive the field towards the global minimum at \( \Phi = \pm \sigma \).
Constraints stemming from the magnitude of density perturbations produced during inflation requires the coupling constant which determines the flatness of \( V(\Phi) \) to be so small that the initial condition \( \Phi(x) \sim 0 \) can no longer be justified. This is taken into account in the chaotic inflationary universe. In this case, either potential (curve a or b of Fig. 6.6) can be chosen. The part of the universe inside our horizon is assumed to stem from a region in space where initially \( \Phi(x) \) was homogeneous over a few Hubble volumes and in addition \( \Phi \gg M_P \) (where \( M_P \) stands for the Planck mass). In this case, \( \Phi(x) \) will slowly roll towards \( \Phi = 0 \), its motion being damped by the expansion of the universe. Again, \( P_\Phi = -\rho_\Phi \), and inflation occurs.

**Generic Inflation Dynamics**

The Friedmann equation in the inflationary phase can then be written as (in units where \( \hbar = 1 = c \), flat model), \( \Phi = \Phi(t) \),

\[
H^2(t) = \frac{8\pi}{3M_P^2} \left[ V(\Phi) + \frac{1}{2} \dot{\Phi}^2 \right].
\]  

(6.13)

In the following, we often use the reduced Planck mass defined as \( M_{Pl} = M_P/\sqrt{8\pi} = 2.4 \times 10^{18} \text{ GeV}/c^2 \), which is a convenient energy scale. The equation of motion for the scalar field reduces to

\[
\ddot{\Phi} + 3H\dot{\Phi} = -\frac{dV}{d\Phi}.
\]  

(6.14)

The two equations can be integrated to give the expansion factor \( a(t) \) and \( \Phi(t) \). The actual solutions will depend on the particular form of the effective potential as well as on the initial conditions. In the limit that \( \dot{\Phi} \to 0 \), the expansion is of the deSitter form, with the scale factor increasing exponentially in time, \( a \propto \exp(Ht) \),

\[
H = \sqrt{\frac{8\pi}{3M_{Pl}^2} V(\Phi)} = \text{const}.
\]  

(6.15)

In general, the Hubble parameter will not be exactly constant, but will vary as the field evolves along the potential \( V(\Phi) \). From the above we get a condition on the kinetic energy term

\[
\frac{1}{2} \dot{\Phi}^2 \sim \frac{1}{2} \left( \frac{V'}{3H} \right)^2 \simeq \frac{|V'|^2}{48\pi V^2/M_{Pl}^2} \ll V.
\]  

(6.16)

The slow rolling condition requires

\[
\frac{(V')^2 M_{Pl}^2}{V} \ll 48\pi.
\]  

(6.17)

From the condition that

\[
3H\dot{\Phi} \simeq -V'
\]  

(6.18)

we obtain by time–derivative

\[
3\dot{H} \dot{\Phi} + 3H\ddot{\Phi} \simeq -V''\dot{\Phi} = \frac{V''V'}{3H}.
\]  

(6.19)
On the other hand, the Friedmann equation implies
\[ \dot{H}H = \frac{4\pi G}{3} \left[ \dot{\Phi} \dot{\Phi} + V' \dot{\Phi} \right] \simeq \frac{4\pi G}{3} V' \dot{\Phi}, \] (6.20)
and therefore
\[ 3\dot{H} \dot{\Phi} = 4\pi \left( \frac{V' \dot{\Phi}^2}{H M_P} \right) \ll 3V' H \] (6.21)
This together requires
\[ 3\dot{H} \dot{\Phi} + 3H \ddot{\Phi} \ll 12HV' \simeq V'' \dot{\Phi} \simeq \frac{V''}{3H}. \] (6.22)
If we replace $H$ from the Friedmann equation, we get a second constraint on the potential
\[ \left| \frac{V''}{M_P^2} \right| \ll 96\pi. \] (6.23)
We need a potential which is very smooth, before it turns to the global minimum.

A convenient approach to the more general case is to express the Hubble parameter directly as a function of the field $\Phi$ instead as a function of time, $H = H(\Phi)$. This is consistent as long as $\Phi$ is monotonic in time. The underlying FRW model can be described as a spatially flat metric
\[ ds^2 = L^2(t) dt^2 - \exp(2\alpha(t))(dx^2 + dy^2 + dz^2), \] (6.24)
where $L(t)$ represents the lapse function and $\alpha(t) = \exp(\alpha(t))$ is the expansion factor. The action for inflationary models is then
\[ S = -\int d^4x \sqrt{-g} \left[ \frac{M_P^2}{16\pi} R - \frac{1}{2} (\nabla \Phi)^2 + V(\Phi) \right], \] (6.25)
where $\mathcal{R}$ is the Ricci scalar. Substitution of the metric ansatz leads to an ADM action of the form

$$S = \int dt \, U L \exp(3\alpha) \left[ -\frac{3M_p^2}{8\pi} \frac{\dot{\alpha}^2}{L^2} + \frac{1}{2} \frac{\dot{\Phi}^2}{L^2} - V(\Phi) \right]$$

(6.26)

with $U = \int d^3x$ as the comoving volume of the universe, which can be normalized to unity.

We consider the scalar field as a dynamical variable of the system. This allows then to write the Einstein–scalar field equations as a set of first order differential equations. For this one considers the Hamiltonian constraint $\mathcal{H} = 0$ by functionally deriving the above action with respect to the non–dynamical lapse function (gauge condition). In this way one arrives at the Hamilton–Jacobi equation

$$-\frac{4\pi}{3M_P^2} \left( \frac{\partial S}{\partial \alpha} \right)^2 + \left( \frac{\partial S}{\partial \Phi} \right)^2 + 2 \exp(6\alpha) V(\Phi) = 0.$$  

(6.27)

$p_\alpha = \frac{\partial S}{\partial \alpha} = -3M_P^2 \exp(3\alpha) \dot{\alpha}/4\pi L$ is the momentum conjugate to $\alpha$, and $p_\Phi = \frac{\partial S}{\partial \Phi} = \exp(3\alpha) \dot{\Phi}/L$ conjugate to $\Phi$. The classical dynamics of this model follows then from the real separable solution

$$S = -\frac{M_P^2}{4\pi} \exp(3\alpha) H(\Phi),$$

(6.28)

where $H(\Phi)$ satisfies the differential equation

$$[H'(\Phi)]^2 - \frac{12\pi}{M_P^2} H^2(\Phi) = -\frac{32\pi^2}{M_P^4} V(\Phi).$$

(6.29)

In the gauge $L = 1$, substitution of the ansatz for the action $S$ into the conjugate momentum gives

$$H(\Phi) = \dot{\alpha}, \quad -\frac{M_P^2}{4\pi} H'(\Phi) = \dot{\Phi}.$$  

(6.30)

Thus, $H(\Phi)$ represents the Hubble expansion parameter expressed as a function of the scalar field. It follows from this that $\dot{H} < 0$, i.e. the physical Hubble radius increases with time as the inflaton field rolls down its potential.

**Potential Reconstruction:** For a given expansion law we can construct the form of the potential. For this let us consider a scalar field with energy and density related over an EOS of the form $w(t) = P(\Phi(t))/\rho(\Phi(t))$. For given $w(t)$ we can integrate the energy conservation law $d(a^3 \rho) = -w \rho da^3$ to obtain

$$\dot{\rho}_\Phi = -3H(1 + w) \rho_\Phi, \quad \rho_\Phi \propto \exp\left[ -3 \int dt(1 + w)H \right].$$

(6.31)

On the other hand $\rho_\Phi \propto H^2$, so that $\dot{\rho}_\Phi = 2(\dot{H}/H)\rho_\Phi$. Combining the two relations we obtain

$$1 + w(t) = -\frac{2}{3} \frac{\dot{H}}{H^2}.$$  

(6.32)
On the other hand, we can also calculate $V(t)$: the definition of $w$ results in $\dot{\Phi}^{2}/2V = (1 + w)/(1 - w) \equiv f(t)$. Writing this as $\dot{\Phi}^{2} = 2fV$ and differentiating, we find

$$\frac{\dot{V}}{V} = -\frac{\dot{f} + 6fH}{1 + f}$$

and this can be integrated to give the solution

$$V(t) = \frac{3H^{2}}{8\pi G} \left[ 1 + \frac{\dot{H}}{3H^{2}} \right] .$$

From the expression $\dot{\Phi}^{2} = 2fV$ we can determine $\Phi(t)$ as

$$\Phi(t) = \int dt \sqrt{-\dot{H}/(4\pi G)} .$$

In this way, we have solved the problem to determine a potential $V$ that will lead to a given expansion $a(t)$. Let us consider the example $a(t) = a_{0}t^{p}$. The potential must then have the form

$$V(\Phi) = V_{0} \exp \left( -\frac{\sqrt{2}}{p M_{Pl}} \Phi \right) .$$

The corresponding evolution of the field is given by

$$\Phi(t) = \sqrt{2}pM_{Pl} \ln \left( \sqrt{\frac{V_{0}}{p(3p - 1)}} \frac{t}{M_{Pl}} \right) .$$

For an exponential expansion $a(t) \propto \exp(At^{q})$ with $q = b/(4 + b)$, $0 < q < 1$, we obtain the solution

$$V(\Phi) \propto \left( \frac{\Phi}{M_{Pl}} \right)^{-b} \left( 1 - \frac{b^{2} M_{Pl}^{2}}{6 \Phi^{2}} \right) .$$

These examples demonstrate that it is possible to have a rapid growth for $a(t)$, provided there is a suitable choice for the potential. This is in particular the case for a slow rollover. Such potentials have a gently decreasing for the potential. Then we can ignore $\ddot{\Phi}$ in the equation of motion. Similarly, the kinetic term given by $\dot{\Phi}^{2}$ can be ignored with respect to the potential energy $V(\Phi)$. In this limit, we simply have

$$H^{2} = \frac{V(\Phi)}{3M_{Pl}^{2}} ,$$

$$3H\dot{\Phi} = -V'(\Phi) .$$

**Growth Factor**

The rollover period is valid until $t_{f}$. The amount of inflation is then characterized by $a(t_{f})/a(t)$. Because this is a large number in most models, it is convenient to work with a number $N(t) \equiv \ln[a(t_{f})/a(t)]$. From the above expressions, we get

$$N \equiv \ln \left( \frac{a(t_{f})}{a(t)} \right) = \int_{t}^{t_{f}} H dt .$$
This can be written as
\[ H \, dt = H \frac{dt}{d\Phi} \, d\Phi = -3H^2 \frac{d\Phi}{V'} . \tag{6.42} \]

The growth factor follows therefore from
\[ N(\Phi_i, \Phi_f) = -3 \int_{\Phi_i}^{\Phi_f} \frac{H^2}{V'} \, d\Phi = -\frac{8\pi}{M_P^2} \int_{\Phi_i}^{\Phi_f} \frac{V}{V'} \, d\Phi \approx \frac{8\pi}{M_P^2} \Phi_i^2 . \tag{6.43} \]

The initial value \( \Phi_i \) has to be chosen in such a way that \( N \gg 1 \), depending on the curvature of the potential \( V' \). The last equality follows from a potential of the typical form \( V(\Phi) = \lambda \Phi^\nu \).

For efficient inflation \( N > 50 \) is required.

The slow rollover approximation requires that the parameters
\[ \epsilon = \frac{M_P^2}{16\pi} \left( \frac{V'}{V} \right)^2 \leq 1 , \quad \eta = \frac{M_P^2}{8\pi} \left| \frac{V''}{V} \right| \leq 1 \tag{6.44} \]
be small numbers.

**New Inflationary Model:** The new inflationary model is based on the following form of the effective potential \( V(\Phi, T) \), taken originally from the work of S. Coleman and E. Weinberg
\[ V(\Phi, T) = \frac{25}{16} \alpha^2 \left[ \Phi^4 \ln(\Phi^2/\sigma^2) + \frac{1}{2}(\sigma^4 - \Phi^4) \right] + \frac{18}{\pi^2} T^4 \int_0^\infty \ln \left[ 1 - \exp \left( -\sqrt{x^2 + 5\Phi^2 g^2/12T^2} \right) \right] dx . \tag{6.45} \]
\( \alpha, g \) and \( \sigma \) are constants. This potential has a false vacuum at \( \Phi = 0 \) and a true vacuum at \( \Phi = \sigma \). There is a temperature dependent bump beyond the false vacuum, followed by a plateau that slopes down very gently before dropping steeply to the true vacuum. Inflation takes place during the time when it tunnels through the bump and then slowly rolls over in the plateau region. When it drops to the true vacuum, the field executes damped oscillations, during which energy is thermalized and entropy increased. The actual dynamics of the Universe can be solved numerically.

**Chaotic Inflation:** Chaotic inflation is given by the potential \( V(\Phi) = \lambda \Phi^4 \) with \( \Phi \) evolving from some initial value \( \Phi_i \) with \( a = a_i \) towards \( \Phi = 0 \) (Fig. 6.7). The slow rollover solution is given by
\[ \Phi(t) = \Phi_i \exp \left[ -\sqrt{2\lambda M_P^2}/(3\pi \, (t - t_i)) \right] \tag{6.46} \]
and the expansion law
\[ a(t) = a_i \exp \left[ \frac{\pi \Phi_i^2}{M_P^2} \left[ 1 - \exp \left\{ -\sqrt{8\lambda M_P^2}/(3\pi \, (t - t_i)) \right\} \right] \right] . \tag{6.47} \]

The growth factor is then
\[ N = \frac{\pi \Phi_i^2}{M_P^2} . \tag{6.48} \]
Resolution of the Horizon Problem: The particle horizon during inflation, given by

\[ d_H(t) = \exp(Ht) \int_{t_i}^{t} \frac{dt'}{\exp(Ht')} \simeq \frac{c}{H} \exp(H(t - t_i)) \]  

(6.49)

for \( t - t_i \gg 1/H \) grows as fast as \( a(t) \) (Fig. 6.5). At the end of inflation \( (t = t_f) \), \( \Phi \) starts oscillating about the minimum of the potential at \( \Phi_0 \). It finally decays and reheats the universe at a temperature \( kT_r \simeq 10^{10} \) GeV. The universe returns then to the normal Big Bang. The horizon \( d_H(t_f) \) is stretched during the oscillations by a factor of \( 10^{9} \), depending on the details, and between \( T_r \) and the present by a factor \( T_r/T_0 \). So it finally becomes equal to \( 10^9(T_r/T_0)(c/H) \exp(H\tau) \), which should exceed \( 2c/H_0 \) in order to solve the horizon problem. With \( V_0 \simeq M_X^4 \) and \( M_X \simeq 10^{16} \) GeV, we see that with \( N = H\tau \geq 55 \), the horizon problem is evaded.

Resolution of the Flatness Problem: The curvature term in the Friedmann equation, at present, is given by

\[ \frac{k}{a_0^2} \simeq \left( \frac{k}{a^2} \right)_{bi} \exp(-2H\tau) 10^{-18} \left( \frac{10^{-13} \text{GeV}}{10^9 \text{GeV}} \right)^2. \]  

(6.50)

The terms of the rhs correspond to the curvature before inflation, and its growth factors during inflation, during \( \Phi \)-oscillations and after reheating, respectively. Assuming \( (k/a^2)_{bi} \simeq (8\pi G/3)\rho \simeq H^2 (\rho \simeq V_0) \), we obtain \( \Omega - 1 = k/a_0^2 H_0^2 \simeq 10^{48} \exp(-2H\tau) \), which is \( \ll 1 \) for \( H\tau \gg 55 \). This is called weak inflation.

Resolution of the Monopole Problem: For \( N > 55 \), the monopoles are diluted by at least 70 orders of magnitude and become irrelevant. There is also no monopole production after reheating.

6.3 The Universe as the Decay Product of a Regular Initial State

In the following we construct a model of the early Universe where the Universe begins from a meta–stable Einstein static state which decays over inflation into a deSitter phase and subsequently evolves into the standard hot Big Bang. The hope is that such a mechanism can be realized in some quantum cosmological framework or some higher dimensional non–singular gravity theory.

The idea that the Universe we inhabit in is in eternal inflation is not new. Arguments were put forward for several inflationary cosmologies that were past–eternal while avoiding a quantum gravity regime. The eternal emergent universe is topologically equivalent of \( \mathbb{R} \times S^3 \) in the past asymptotic region having an initial radius \( a_0 \gg \Lambda_P \) [6]. One would expect a behaviour of the scale factor of the form \( a(t) = a_0[1 + \exp(Ht)] \). The construction of the corresponding potential has the desired early behaviour, but does not show a definitely zero minimum. We need to find a universe with a potential that is essentially the same as above at very early times, but then goes to zero at some finite value of the field, i.e. the potential should have a long flat plateau for \( t \rightarrow -\infty \), but then has a vanishing minimum value (Fig. 6.8). Such
6.3 The Universe as the Decay Product of a Regular Initial State

a behaviour in fact occurs in the models considered by Starobinsky [24], in which the deSitter phase was driven by the trace anomaly of the energy momentum tensor, the corresponding action is based on quadratic terms in the Lagrangian, \( \mathcal{L} = R + \alpha R^2 \) with the Ricci scalar \( R \). One can then show that this is equivalent to an inflationary model with a scalar field related to the quadratic terms, \( \Phi \equiv \sqrt{3} \ln[1 + 2\alpha R] \)

\[
S = \int \sqrt{-\bar{g}} \, d^4x \left[ \bar{R} - \frac{1}{2} \partial_{\mu} \Phi \partial^{\mu} \Phi - \frac{1}{4\alpha} \left[ \exp(-\Phi/\sqrt{3}) - 1 \right]^2 \right]. \tag{6.51}
\]

Hereby, the metric is conformally transformed by \( \Omega^2 = 1 + 2\alpha R \) providing a new metric \( \bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \) and the volume \( \sqrt{-\bar{g}} = \Omega^4 \sqrt{-g} \).

We now abstract from the above considerations and just postulate a potential of the form as described by the above quantum gravity corrections to the classical action. These models are **spatially closed**. Emergent universes can be realized by considering a spacetime filled with a self–interacting scalar field with a potential of the form

\[
V(\Phi) = \left[ A \exp(B\Phi) - C \right]^2 + D. \tag{6.52}
\]

Then

\[
V' = 2AB \left[ A \exp(B\Phi) - C \right] \exp(B\Phi) \tag{6.53}
\]

and

\[
V'' = 2AB^2 \left[ 2A \exp(B\Phi) - C \right] \exp(B\Phi). \tag{6.54}
\]

This potential has a minimum at \( \Phi_0 = (1/B) \ln(C/A) \) with \( V(\Phi_0) = D \). If we want to set the minimum at the origin of the axes we need to satisfy \( D = 0 \) and \( A = C \). Therefore, the

**Figure 6.8:** Potential of the Emergent Universe. The minimum occurs at the value \( \Phi = 0 \). After its initial static phase (1), the model enters a slow–rolling phase (2–3). The expansion factor grows sufficiently rapidly making curvature effects meaningless. After the reheating phase (4) the universe enters the standard Big Bang expansion.
potential is given by two parameters

\[ V(\Phi) = A \left[ \exp(B\Phi) - 1 \right]^2 . \] (6.55)

For \( \Phi \to -\infty \) we get an Einstein static universe with

\[ V(\Phi = -\infty) = \frac{2}{\kappa a_0^2} . \] (6.56)

It is then clear that for \( a_0 \gg \Lambda_P \), the model can avoid a quantum regime. So the potential has now the form

\[ V(\Phi) = \frac{2}{\kappa a_0^2} \left[ \exp(B\Phi) - 1 \right]^2 \] (6.57)

and is given by two parameters, the initial radius \( a_0 \) and the parameter \( B \). The remaining parameter \( B \) will be determined by other properties, in particular by looking into the density perturbation. The sign of \( B \) is determined by the condition \( V' < 0 \) for \( -\infty < \Phi < 0 \). This means that \( \exp(B\Phi) - 1 < 0 \), and therefore \( B > 0 \).

As the universe leaves the Einstein–static state, the evolution is dictated by the Klein–Gordon equation (6.14). Since the potential drops with increasing \( \Phi \), it may evolve in a number of ways depending on the relation between \( \dot{\Phi} \) and \( V'' \). As discussed above, the duration of the slow–roll phase is then determined by the slow–roll parameters

\[ \epsilon = \frac{1}{2} M_P^2 \left( \frac{V'}{V} \right)^2 , \quad \eta = M_P^2 \frac{V''}{V} . \] (6.58)

Applied to our potential, the above conditions read

\[ \epsilon(\Phi) = \frac{2B^2 M_P^2 \exp(2B\Phi)}{[\exp(B\Phi) - 1]^2} \ll 1 \] (6.59)

and

\[ \eta(\Phi) = \frac{2B^2 M_P^2 [2 \exp(B\Phi) - 1] \exp(B\Phi)}{[\exp(B\Phi) - 1]^2} \ll 1 . \] (6.60)

Though this model is closed, curvature effects become negligible already after a few e–foldings, they will emerge only in the recent universe! Slow–roll then begins at an arbitrarily small value of \( \Phi \).

For \( B > 0 \) there is always a negative value of \( \Phi \) at which the slow–roll phase ends, i.e. where \( \epsilon \approx 1 \approx \eta \). For example when \( B = 1/M_P \), then one finds \( \Phi < -M_P \ln[1 + \sqrt{2}] \).

**Number of e–Foldings:** The number of e–foldings is given by

\[ N = -\frac{\kappa}{2B} \int_{\Phi_i}^{\Phi_f} \frac{\exp(B\phi) - 1}{\exp(B\Phi)} \, d\Phi . \] (6.61)

This can easily be computed numerically (Fig. 6.9). Clearly, depending on the value of \( \Phi_i \), sufficiently e–folds are obtained in this model for \( BM_P \approx 1 \).
6.3 The Universe as the Decay Product of a Regular Initial State

Figure 6.9: Number of $e$–foldings in the Emergent Universe plotted against the initial value $\Phi_i$ in units of $M_P$ and the parameters $B$ in units of $1/M_P$.

Scale Factor Evolution: Starting from the Friedmann equation

$$H^2 + \frac{1}{a^2} = \frac{1}{3} \kappa \left( \frac{1}{2} \dot{\Phi}^2 + V \right)$$

we get in the slow–roll approximation

$$\frac{\dot{\Phi}}{a} \frac{da}{d\Phi} = \sqrt{\frac{1}{3} \kappa V}.$$  \hspace{1cm} (6.63)

Since in this phase $\ddot{\Phi} \simeq 0$, the Klein–Gordon equation gives $\dot{\Phi} \simeq -V'/\sqrt{3\kappa V}$. Combining the two equations we find

$$\frac{1}{a} \frac{da}{d\Phi} = -\frac{\kappa V}{V''} d\Phi = \frac{\kappa}{2B} \frac{1 - \exp(B\Phi)}{\exp(B\Phi)} d\Phi,$$  \hspace{1cm} (6.64)

which can be integrated to yield

$$\ln \left( \frac{a(t)}{a_i} \right) = -\frac{\kappa}{2B^2} \left[ \exp(-B\Phi) - \exp(-B\Phi_i) \right] - \frac{\kappa}{2B} (\Phi - \Phi_i).$$  \hspace{1cm} (6.65)

This shows then that the smaller $\Phi_i$, the larger the final value of $a$ (Fig. 6.10).

The characteristic \textbf{evolution time} follows from the Klein–Gordon equation

$$\dot{\Phi} \simeq -\frac{V'}{\sqrt{3\kappa V}},$$  \hspace{1cm} (6.66)

which implies

$$t_{ev} \simeq \frac{\Phi_i}{d\Phi/dt} \simeq \frac{2\pi \sqrt{6} a_0 \Phi_i}{BM_P^2} \exp(-B\Phi_i),$$  \hspace{1cm} (6.67)
or in natural units

\[ t_{ev} \simeq \frac{2\pi \sqrt{6} a_0}{c} \frac{\Phi_i}{M_P(BM_P)} \exp[\frac{-(BM_P)(\Phi_i/M_P)}{BMP}] . \] (6.68)

For a standard value of \( a_0 \simeq 10^6 \Lambda_P \) (see below), the inflationary phase occurs on a time scale \( t_{ev} \simeq 10^{-34} \) sec, as required by standard arguments (for parameters \( B \simeq 1/M_P \) and \( \Phi_i \simeq -4M_P \)).

**The Perturbation Spectrum:** Following the above discussion, the perturbation spectrum follows from

\[ \left( \frac{\delta \rho}{\rho} \right)_{\text{Hor}} \simeq \left( \frac{H^2}{\Phi} \right)_{N=50} \simeq \left( \frac{3H^2}{V'} \right)_{N=50} . \] (6.69)

Using the slow–roll approximation this provides

\[ \left( \frac{\delta \rho}{\rho} \right)_{\text{Hor}} \simeq -\sqrt{3} \kappa \left( \frac{V^{3/2}}{V'} \right)_{N=50} \] (6.70)

With the expressions of our potential this gives

\[ \left( \frac{\delta \rho}{\rho} \right)_{\text{Hor}} \simeq \frac{\kappa}{\sqrt{6}Ba_0} \left[ \frac{\exp(B\Phi) - 1]^2}{\exp(B\Phi)} \right]_{N=50} \simeq 10^{-5} . \] (6.71)
6.4 Density Perturbations from Inflation

The CMBR conditions give us therefore a relation between the parameter $B$ and the radius $a_0$. For example, when $B = 1/M_P$, then the perturbation requirement leads to $a_0 \simeq 10^6 \Lambda_P$. This is an interesting result, since it would mean that quantum gravity is not important for the initial state!

Also the power spectrum can be estimated from these considerations, but a rigorous calculation based on quantum field theory considerations is still missing.

Available Energy for Reheating: In this model inflation starts at $V(\Phi \rightarrow -\infty) = 2/\kappa a_0^2$, this means that the maximum energy stored in the inflaton field is $V(\Phi \rightarrow \infty) = 2/\kappa a_0^2 \simeq 10^{37}/a_0^2 \text{ GeV}^2$. The maximum energy available for reheating is proportional to $1/a_0^2$, i.e.

$$V(\Phi \rightarrow \infty) \simeq 10^{63} \text{ GeV}^4 (10^6 \Lambda_P/a_0)^2.$$ (6.72)

Therefore, provided that thermal equilibrium can be achieved and that the reheating process is efficient, the temperature at the beginning of the standard Big Bang would be as high as

$$k_B T_{RH} \simeq 10^{16} \text{ GeV}.$$ (6.73)

For less efficient reheating, the temperature can be lower.

Final Remarks: Models of this type show for the first time that the initial singularity can be avoided, though some fine-tuning is still necessary. The Einstein static universe has certainly the highest symmetry. Such an initial state might result from transitions in some higher dimensional theory (as a kind of gravitational instanton), where self-gravity can be modelled in a non-singular way. For me, it is a very appealing aspect of this model that the present Universe is the result of an initial state slowly evolving from past infinity, instead of starting from a singular state at a finite time.

6.4 Density Perturbations from Inflation

Very soon after the introduction of inflation by Guth, it was realized that inflation had another remarkable property: it could explain the generation of the primordial density fluctuations in the universe. This was first worked out independently by Hawking [11], Starobinsky [24], and by Guth [9]. The generation of fluctuations is covered in details below, but the basic physics is familiar, and is closely related to the generation of Hawking radiation by black holes. It can be explained qualitatively as follows. In a normal Minkowski space, vacuum fluctuations are interpreted as pairs of virtual particles appearing and then immediately annihilating as a consequence of the Heisenberg uncertainty principle. One qualitative explanation of Hawking radiation at the event horizon of a black hole is that one of the two virtual particles is trapped by the horizon, leaving the other to escape as apparently thermal radiation. A similar process holds in an inflationary spacetime: in inflation, the expansion is so rapid that pairs of virtual particles get "swept up" in the spacetime and are inflated to causally disconnected regions. In essence, they can no longer find each other to annihilate, and the quantum fluctuations become classical modes of the field.

Formally, this effect is calculated by considering the equation of motion for a fluctuation in a free field $\delta \Phi$,

$$\ddot{\delta \Phi} + 3H \dot{\delta \Phi} + \left( \frac{k}{a} \right)^2 \delta \Phi = 0$$ (6.74)
where $k$ is a comoving wavenumber which stays constant with expansion. The physical momentum $p$ of the particle is $p = k/a$. During inflation, the wavelength $\lambda$ of a quantum mode is "stretched" by the rapid expansion, $\lambda \propto a(t) \propto \exp(\mathcal{H}t)$. The horizon size, however, remains roughly constant, $d_H = c/H \simeq \text{const}$. Short-wavelength vacuum fluctuations are then quickly redshifted by the expansion until their wavelengths are larger than the horizon size of the spacetime, and the modes are "frozen" as classical fluctuations. The amplitudes of quantum modes in inflation are conventionally expressed at horizon crossing, that is when the wavelength of the mode is equal to the horizon size, or $k/a = \mathcal{H}$. The two-point correlation function of the field at horizon crossing is just given by the Hubble parameter:

$$< \delta \Phi^2 >_{(k/a)_{\text{Hor}}} = \left(\frac{H}{2\pi}\right)^2.$$ (6.75)

Inflation not only homogenizes the universe but also provides us with the primordial density perturbations needed for structure formation. The event horizon at cosmic time $t$ includes all points with which we will eventually communicate by sending signals at $t$

$$d_E(t) = a(t) \int_{t'}^{\infty} \frac{dt'}{a(t')}.$$ (6.76)

For inflation we obtain a more or less constant event horizon with $d_E = c/H < \infty$. Points in our event horizon at time $t$, with which we can communicate by sending signals, are eventually pulled away by the exponential expansion and we cease to communicate with them. We say that these points crossed the horizon. The situation is similar to that of a Black Hole. Indeed, the exponentially expanding deSitter space is like a Black Hole turned inside out. We are inside and the Black Hole surrounds us from all sides. Then, exactly as in the Black Hole case, there are quantum fluctuations of the thermal type governed by the Hawking temperature. The presence of this event horizon suggests the presence of thermal fluctuations in the fields, similar to those present in a black hole. This can be understood by appealing to the uncertainty principle. The event horizon causes the ground state modes of any fields present to be restricted in spatial extent. The uncertainty principle then requires, $\Delta p \geq \hbar \mathcal{H}/c$, since the characteristic size is just $c/H \simeq 10^{-24}$ cm, which is much bigger than the Planck scale!

This uncertainty in momentum gives rise to energy fluctuations and the corresponding Hawking temperature is given by (like the horizon of a black hole, the horizon of an inflationary space has a "temperature")

$$k_B T_{\text{deSitter}} = \frac{\hbar H}{2\pi} \simeq 10^{22} K \simeq 10^9 \text{GeV},$$ (6.77)

provided $1/H \simeq 10^{-34}$ sec. This result provides a motivation for the existence of fluctuations in the metric and scalar field. As these perturbations are created during inflation they are inflated outside of the causal horizon (event horizon). As mentioned in the previous section, the causal horizon is nearly stationary during inflation. Once the perturbation has been inflated outside the horizon, its ends are no longer in causal contact. In this way, the perturbations become "frozen-in" as classical perturbations.

The generation of scalar fluctuations is quite complex, since scalar modes are generated by fluctuations in the inflaton field itself and couple to the curvature of the spacetime

$$\left(\frac{\delta \rho}{\rho}\right)_{k=aH} = \frac{1}{2\sqrt{\pi}} \frac{\delta N}{\delta \Phi} \frac{\delta \Phi}{H^2}.$$

(6.78)
where \( N \) is the number of e-folds of inflation. These are the amplitudes for a single mode when its wavelength (which is changing with time due to expansion) is equal to the horizon size. In the case of slow roll, with \( \dot{\Phi} \) small and \( H \) slowly varying, modes of different wavelengths will have approximately the same amplitudes, with slow variation as a function of scale. If we define the power spectrum as the variance per logarithmic interval,

\[
\left( \frac{\delta \rho}{\rho} \right)^2 = \int P_S(k) \, d\ln k
\]

(6.79)

inflation generically predicts a power-law form for \( P_S(k) \), \( P_S(k) \propto k^{n-1} \), so that the scale invariant spectrum, one with equal amplitudes at horizon crossing, is given by \( n = 1 \). The current observational best fit for the spectral index \( n \) is \( n = 0.91 \pm 0.1 \) (the best value obtained from first–year WMAP is \( n = 0.97 \pm 0.03 \)).

**Example: Chaotic Inflation**

Let us consider the potential \( V(\Phi) = \lambda \Phi^4 \). We will assume up front that the field is slowly rolling

\[
\dot{\Phi} = -\frac{V'}{3H} = -\sqrt{\frac{M_P^2}{24\pi}} \frac{V'}{\sqrt{V(\Phi)}}.
\]

(6.80)

In order for inflation to occur, we must have negative pressure, \( P < -1/3 \), which is equivalent to the slow roll parameter \( \epsilon \) being less than unity,

\[
\epsilon = \frac{M_P^2}{16\pi} \left( \frac{V'}{V(\Phi)} \right)^2 = \frac{1}{\pi} \left( \frac{M_P}{\Phi} \right)^2 < 1.
\]

(6.81)

Inflation occurs when \( \Phi > \Phi_f = (M_P/\sqrt{\pi}) \). Note that the field is displaced a long way from the minimum of the potential at \( \Phi = 0 \)! This has been the source of some criticism of this type of model as a valid potential in an effective field theory, but here we will accept this fact at the very least as valid phenomenology.

In this simple model, then, we have inflation happening when the field is rolling down the potential in a region far displaced from the minimum \( \Phi > M_P \). Inflation ends naturally at late time, when \( \Phi \) passes through \( \Phi_f = M_P/\sqrt{\pi} \). In order to solve the horizon and flatness problems, we must have at least a factor of \( e^{55} \) expansion. The number of e–folds is given by

\[
N = \int H \, dt = \int \frac{H}{\dot{\Phi}} \, d\Phi.
\]

(6.82)

It is convenient to choose the limits on the integral such that \( N = 0 \) at the end of inflation, so that \( N \) counts the number of e-folds until inflation ends and increases as we go backward in time. Then, using the equation of motion for the field, we can show that \( N \) is just an integral over the slow-roll parameter \( \epsilon \), and can be expressed as a function of the field value \( \Phi \)

\[
N(\Phi) = \frac{2\sqrt{\pi}}{M_P} \int_{\Phi_f}^{\Phi} \frac{d\Phi'}{\sqrt{\epsilon(\Phi')}}.
\]

(6.83)

For our potential, the number of e–folds is

\[
N(\Phi) = \frac{\pi}{M_P} (\Phi - \Phi_f)^2 = \pi \left( \frac{\Phi}{M_P} \right)^2 - 1.
\]

(6.84)
We now want to evaluate the power spectrum amplitude $P_S$, and scalar spectral index $n_S$ for fluctuations with scales comparable to the horizon size today, which means fluctuations which crossed outside the horizon during inflation at about $N \simeq 55$. Therefore, to calculate the amplitude $P_S$, we evaluate

$$\sqrt{P_S} = \frac{H^2}{4\pi^{3/2} \Phi} = 4\sqrt{\frac{2}{3}} \frac{[V(\Phi)]^{3/2}}{M_P^3 V'(\Phi)}\quad \text{at} \; \Phi_{55},$$

i.e.

$$\sqrt{P_S} = \sqrt{\frac{2\lambda}{3}} \left( \frac{\Phi_{55}}{M_P} \right)^3 \simeq 60 \sqrt{\lambda}. \quad \text{(6.86)}$$

But from the CMBR, we know that the power spectrum amplitude is $P_S^{1/2} \simeq 10^{-5}$, so that means we must have a very tiny self-coupling for the field, $\lambda \simeq 10^{-14}$. In order to sufficiently suppress the density fluctuation amplitude, the model must be extremely fine–tuned. This is a typical property of scalar field models of inflation.

This also allows us to estimate the energy scale of inflation,

$$E \simeq V^{1/4}(\Phi) \simeq \lambda^{1/4} \Phi_{55} \simeq 10^{-3} M_P \simeq 10^{16} \text{GeV}. \quad \text{(6.87)}$$

Finally, it is straightforward to calculate the spectral index \[ n_S = 1 - 4\epsilon(\Phi_{55}) + 2\eta(\Phi_{55}) = 1 - \frac{3}{56} = 0.95. \quad \text{(6.88)} \]

The procedure for other potentials is similar: first, find the field value where inflation ends. Then calculate the field value 55 e-folds before the end of inflation and evaluate the expressions for the observables at that field value. In this way we can match any given model of inflation to its observational predictions. In the next section, we examine the predictions of different types of models in light of current and future observational constraints, and find that it will be possible with realistic measurements to distinguish between different models of inflation.

### A Rigorous Formula

It turns out that the quantum fluctuations of all massless fields are $\delta \Phi = k_B T_H$ (and the inflaton is a massless field). These fluctuations lead to energy density perturbations $\delta \rho = V'(\Phi) \delta \Phi$. As the scale of these perturbations crosses outside the event horizon, they become classical metric perturbations.

As we will show in Sect. 7, there is a simple gauge–invariant quantity $\zeta \simeq \delta \rho/(\rho + P)$ which remains constant outside the horizon. Thus the density perturbations $(\delta \rho/\rho)_\lambda$ at the scale $\lambda$, when it crosses the particle horizon, can be related to its value when the same scale crossed outside the event horizon. This latter value is found from the inflationary equation

$$3H \dot{\Phi} = -V'(\Phi)$$

by means of

$$\zeta_{\lambda \simeq c/H} = \left( \frac{V'(\Phi) H(\Phi)}{2\pi \Phi^2} \right)_{\lambda \simeq c/H} = - \left( \frac{9 H^3(\Phi)}{2\pi \Phi^2} \right)_{\lambda \simeq c/H}. \quad \text{(6.90)}$$
6.4 Density Perturbations from Inflation

Taking into account an extra factor $2/5$ from the fact that the universe is matter-dominated when the scale $\lambda$ reenters the horizon, we obtain

$$
\left(\frac{\delta \rho}{\rho}\right)_\lambda = \frac{16\sqrt{6\pi}}{5} \frac{V^{3/2}(\Phi)}{M_P^2V'(\Phi)}.
$$

(6.91)

$\Phi_\lambda$ is the value of the inflaton field when the comoving scale $\lambda$ crossed outside the event horizon. A comoving (present) scale $\lambda$ was equal to $\lambda a(t_d)/a(t_0) = \lambda(T_0/T_r)(t_f/t_d)^{(\nu+2)/3\nu} = \lambda_{phys}(t_f)$, for a potential $V(\Phi) = \lambda_\Phi \Phi^\nu$. The scale $\lambda$ when it crossed outside the inflationary horizon, was equal to $c/H(\Phi_\lambda)$. We thus obtain

$$
[c/H(\Phi_\lambda)] \exp(N(\Phi_\lambda)) = \lambda_{phys}(t_f).
$$

(6.92)

$N(\Phi_\lambda) = N_\lambda$ is the number of $e$-foldings the scale $\lambda$ suffered during inflation. In particular for our present $\lambda = 2c/H_0 \simeq 9000$ Mpc, it turns out that $N_{H_0} \simeq 50 - 60$.

For a potential of the form $V = \lambda \Phi^4$, we obtain

$$
\left(\frac{\delta \rho}{\rho}\right)_\lambda = \frac{4\sqrt{6\pi}}{5} \sqrt{\lambda_\Phi} \left(\frac{\Phi_\lambda}{M_P}\right)^3 = \frac{4\sqrt{6\pi}}{5} \sqrt{\lambda_\Phi} \left(\frac{N_\lambda}{\pi}\right)^{3/2}.
$$

(6.93)

From the CMBR $(\delta \rho/\rho)_{H_0} \simeq 6 \times 10^{-5}$, one can deduce that $\lambda_\Phi \simeq 6 \times 10^{-14}$ for $N_{H_0} \simeq 55$. The inflaton must be very weakly coupled. This is not satisfactory. In SUSY GUTs, however, the inflaton could be identified with a conjugate pair of gauge non-singlet fields, causing the gauge symmetry breaking.

Pure exponential inflation, which corresponds to a deSitter spacetime, has an interesting property. The spacetime is invariant under time translation. That is to say, there is no natural origin of time under true exponential expansion. The only fundamental size in the theory is that of the Hubble horizon $c/H$. Thus, one expects that the amplitude of a ‘standing wave’ perturbation will be related to the horizon size, $c/H$, which is not changing. Therefore, we see why inflation predicts a scale-invariant spectrum for the perturbations.

This analysis can be illustrated through musical analogy. The fundamental mode of the perturbations are determined by the Hubble distance ($c/H$), much like the fundamental mode of a flute is determined by its length. Because the Hubble length (horizon) is nearly stationary during inflation, this means inflation predicts a scale-invariant, or Harrison-Zeldovich spectrum. Furthering this analogy, the ‘overtones’ of the universe correspond to the inflaton potential that determines its behavior, much like overtones can be used to distinguish one instrument from another. However, as we have seen, inflation need not be exponential. The small deviations from deSitter spacetime result in small deviations from a scale-invariant spectrum. These deviations can be used to successfully predict the correct potential for the inflaton.

**Quantum Theoretical Considerations**

Quantum theory is well worked within the context of field theory in special relativity. What about in an expanding universe? The generalization to a curved spacetime is straightforward, if a bit mysterious. We will replace the metric for special relativity with a Robertson-Walker metric,

$$
ds^2 = a^2(\tau)[d\tau^2 - d\vec{x}^2].
$$

(6.94)
Note that we have written the Robertson-Walker metric in terms of conformal time $d\tau = dt/a$. This is convenient for doing field theory, because the new spacetime is just a Minkowski space with a time-dependent conformal factor. In fact we define the physical vacuum in a similar way to how we do it for special relativity: the vacuum is the zero-particle state as seen by a geodesic observer, that is, one in free-fall in the expanding space. This is referred to as the Bunch-Davies vacuum.

Now we write down the wave equation for a free field. This is the usual Klein–Gordon equation with a new term that comes from the expansion of the universe

$$\frac{\partial^2 \Phi}{\partial \tau^2} + 2\frac{a'}{a} \frac{\partial \Phi}{\partial \tau} - \nabla^2 \Phi = 0. \quad (6.95)$$

Note that the time derivatives are with respect to the conformal time $\tau$, not the coordinate time $t$. As in the Minkowski case, we Fourier expand the field, but with an extra factor of $a(\tau)$ in the integral:

$$\Phi = \int d^3 x \left[ a_k u_k(\tau) \exp(ik \cdot \vec{x}) + a_k^+ u_k^*(\tau) \exp(-ik \cdot \vec{x}) \right]. \quad (6.96)$$

Here $k$ is a comoving wavenumber (or, equivalently, momentum), which stays constant as the mode redshifts with the expansion $\lambda \propto a$, so that $k_{\text{phys}} = k/a$. The creation and annihilation operators $a_k^+$ and $a_k$ act on $n$–particle states by adding or subtracting a particle from the state.

Writing things this way, in terms of conformal time and comoving wavenumber, makes the equation of motion for the mode $u_k(\tau)$ very similar to the mode equation in Minkowski space

$$u''_k + \left( k^2 - \frac{a''}{a} \right) u_k = 0, \quad (6.97)$$

where a prime denotes a derivative with respect to conformal time. All of the effect of the expansion is in the $a''/a$ term. This equation is easy to solve. First consider the short wavelength limit, that is large wavenumber $k$. For $k^2 \gg a''/a$, the mode equation is just what we have for Minkowski space

$$u''_k + k^2 u_k \approx 0, \quad (6.98)$$

except that we are now working with comoving momentum and conformal time, so the space is only quasi-Minkowski. The general solution for the mode is

$$u_k = A(k) \exp(-ik\tau) + B(k) \exp(i k \tau). \quad (6.99)$$

Here is where the definition of the vacuum comes in. Selecting the Bunch-Davies vacuum is equivalent to setting $A = 1$ and $B = 0$, so that the annihilation operator is multiplied by $\exp(-ik\tau)$ and not some linear combination of positive and negative frequencies. So the mode function corresponding to the zero-particle state for an observer in free fall is $u_k \propto \exp(-ik\tau)$.

In the long wavelength limit, $k^2 \ll a''/a$, the mode equation becomes trivial

$$u''_k - \frac{a''}{a} u_k = 0 \quad (6.100)$$
with the solution
\[ \left| \frac{u_k}{a} \right| = \text{const.} \quad (6.101) \]

The mode is said to be frozen at long wavelengths, since the oscillatory behavior is damped. This is precisely the origin of the density and gravity-wave fluctuations in inflation. Modes at short wavelengths are rapidly redshifted by the inflationary expansion so that the wavelength of the mode is larger than the horizon size. We can plot the mode as a function of its physical wavelength \( \lambda = k/a \) divided by the horizon size \( d_H = c/H \) (Fig. 6.11), and find that at long wavelengths, the mode freezes out to a nonzero value. The power spectrum of fluctuations is just given by the two-point correlation function of the field,
\[ P(k) \propto \left< \Phi^2 \right>_{(k/a \gg aH)} \propto \left| \frac{u_k}{a} \right|^2 \neq 0. \quad (6.102) \]

This means that we have produced classical perturbations at long wavelength from quantum fluctuations at short wavelength.

**Importance of Planck-Scales:** For processes probing length scales shorter than Planck-scale, such as quantum modes with wavelengths \( < \Lambda_P \), we expect some sort of new physics

![The Mode Function](image_url)

**Figure 6.11:** The mode function \( u_k/a \) as a function of \( d_H/\lambda = k/(aH) = -k\tau \). At short wavelengths, \( k \gg aH \), the mode is oscillatory, but freezes out to a nonzero value at long wavelengths, \( k \ll aH \).
to be important. There are a number of ideas for what that new physics might be, for example string theory or noncommutative geometry or discrete spacetime, but physics at the Planck scale is currently not well understood. It is unlikely that particle accelerators will provide insight into such high energy scales, since quantum modes with wavelengths less than $\Lambda_P$ will be characterized by energies of order $10^{19}$ GeV or so, and current particle accelerators operate at energies around $10^3$ GeV. However, we note an interesting fact, namely that the ratio between the current horizon size of the universe and the Planck length is about

$$d_H/\Lambda_P \simeq 10^{60}, \quad \ln(d_H/\Lambda_P) \simeq 140.$$ (6.103)

This is a big number, but we recall our earlier discussion of the flatness and horizon problems and note that inflation, in order to adequately explain the flatness and homogeneity of the universe, requires the scale factor to increase by at least a factor of $e^{55}$. Typical models of inflation predict much more expansion, $e^{100}$, or more. We remember that the wavelength of quantum modes created during the inflationary expansion, such as those responsible for density and gravitational-wave fluctuations, have wavelengths which redshift proportional to the scale factor, so that the wavelength $\lambda_i$ of a mode at early times can be given in terms of its wavelength $\lambda_0$ today by

$$\lambda_i \ll \lambda_0 \exp(-N).$$ (6.104)

This means that if inflation lasts for more than about $N \simeq 140$ e–folds, fluctuations of order the size of the universe today were smaller than the Planck length during inflation! This suggests the possibility that Plank-scale physics might have been important for the generation of quantum modes in inflation. The effects of such physics might be imprinted in the pattern of cosmological fluctuations we see in the CMB and large-scale structure today. In what follows, we will look at the generation of quantum fluctuations in inflation in detail, and estimate how large the effect of quantum gravity might be on the primordial power spectrum.

We just state that for an inflationary period that lasts longer than 140 e–folds or so, the fluctuations we see with wavelengths comparable to the horizon size today started out with wavelengths shorter than the Planck length $\Lambda_P \simeq 10^{-33}$ cm during inflation. For a mode with a wavelength that short, do we really know how to select the "vacuum" state, which we have assumed is given $\exp(-ik\tau)$? Not necessarily. We do know that once the mode redshifts to a wavelength greater than $\Lambda_P$, it must be of the form (6.101), but we no longer know for certain how to select the values of the constants $A(k)$ and $B(k)$. What we have done is mapped the effect of quantum gravity onto a boundary condition for the mode function $u_k$. In principle, $A(k)$ and $B(k)$ could be anything! If we allow $A$ and $B$ to remain arbitrary, it is simple to calculate the change in the two-point correlation function at long wavelength,

$$P(k) \rightarrow |A(k) + B(k)|^2 P_{DB}(k),$$ (6.105)

where the subscript $BD$ indicates the value for the case of the "standard" Bunch-Davies vacuum, which corresponds to the choice $A = 1, B = 0$. So, the power spectrum of gravity-wave and density fluctuations is sensitive to how we choose the vacuum state at distances shorter than the Planck scale, and is in principle sensitive to quantum gravity.

## 6.5 Reheating after Inflation

During inflation, all matter except the scalar field (which is called the inflaton) is redshifted to extremely low densities. Reheating is the process whereby the inflaton’s energy density is
converted back into conventional matter after inflation. Once the slow–roll conditions break down, the scalar field switches from being overdamped to being underdamped and begins to oscillate at the bottom of the potential. Thereby, it decays into conventional matter. The details of reheating are important for the global understanding, but are of no relevance to the question of the origin of density perturbations. Traditional treatments just add a phenomenological decay term.

6.6 BraneWorld Cosmology – Beyond Dimension Four

The possibility that spacetime has more than 4 dimensions is an old idea. Originally, the extra dimensional spaces were thought to be compactified. This kind of model is usually called Kaluza–Klein compactification. To have a correct Newtonian limit, the size of the extra dimensions must be less than the Electroweak scale (\(\approx 1 \text{ TeV}^{-1}\)). In a setting where the extra dimensions could be large, under the assumption that ordinary matter is confined onto a 3–dimensional subspace, called brane, embedded into a larger space, called bulk. This idea came out of string theory, or M–theory, which describes the low–energy effective theory of an \(E_8 \times E_8\) heterotic string theory. This model is associated with an 11–dimensional supergravity.

In the standard model of particle physics, particles are considered to be points moving through space, tracing out a line called the World Line. To take into account the different interactions observed in Nature one has to provide particles with more degrees of freedom than only their position and velocity, such as mass, electric charge, color (which is the charge associated with the strong interaction) or spin.

In String Theory, the myriad of particle types is replaced by a single fundamental building block, a string. These strings can be closed, like loops, or open, like a hair. As the string moves through time it traces out a tube or a sheet, according to whether it is closed or open. Furthermore, the string is free to vibrate, and different vibrational modes of the string represent the different particle types, since different modes are seen as different masses or spins.

The particles known in nature are classified according to their spin into bosons (integer spin) or fermions (odd half integer spin). The former are the ones that carry forces, for example, the photon, which carries electromagnetic force, the gluon, which carries the strong nuclear force, and the graviton, which carries gravitational force. The latter make up the matter we are made of, like the electron or the quark. The original String Theory only described particles that were bosons, hence Bosonic String Theory. It did not describe Fermions. So quarks and electrons, for instance, were not included in Bosonic String Theory. By introducing Supersymmetry to Bosonic String Theory, we can obtain a new theory that describes both the forces and the matter which make up the Universe. This is the theory of superstrings. There are three different superstring theories which make sense, i.e. display no mathematical inconsistencies. In two of them the fundamental object is a closed string, while in the third, open strings are the building blocks. Furthermore, mixing the best features of the bosonic string and the superstring, we can create two other consistent theories of strings, Heterotic String Theories.

One of the most remarkable predictions of String Theory is that space-time has ten dimensions! At first sight, this may be seen as a reason to dismiss the theory altogether, as we obviously have only three dimensions of space and one of time. However, if we assume that six of these dimensions are curled up very tightly, then we may never be aware of their existence. Furthermore, having these so-called compact dimensions is very beneficial if String Theory is to describe a Theory of Everything. The idea is that degrees of freedom like the electric charge of an electron will then arise simply as motion in the extra compact directions! The principle that compact dimensions may lead to unifying theories is not new, but dates from the 1920’s, since the theory of Kaluza and Klein. In a sense, String Theory is the ultimate Kaluza-Klein theory. For simplicity, it is usually assumed that the extra dimensions are wrapped up on six circles. For realistic results they are treated as being wrapped up on mathematical elaborations known as Calabi-Yau Manifolds and Orbifolds.

Apart from the fact that instead of one there are five different, healthy theories of strings (three superstrings and two heterotic strings) there was another difficulty in studying these theories: we did not have tools to explore the theory over all possible values of the parameters in the theory. Each theory was like a large planet of which we only knew a small island somewhere on the planet. But over the last four years, techniques were developed to explore the theories more thoroughly, in other words, to travel around the seas in each of those planets and find new islands. And only then it was realized that those five string theories are actually islands on the same planet, not different ones! Thus
An important consequence of extra dimensions is that the 4D Planck scale $M_P$ is no longer the fundamental scale, which is now $M_{(4+d)}$, where $d$ is the number of extra dimensions. For this we consider an Einstein–Hilbert gravitational action

$$S = \frac{1}{2\kappa^2_{(4+d)}} \int d^4x \sqrt{-g} \left( (4+d)R - 2\Lambda_{(4+d)} \right)$$

with the corresponding Einstein equation

$$(4+d)G_{AB} = (4+d)R_{AB} - \frac{1}{2}(4+d)Rg_{AB} = \kappa^2_{(4+d)}(4+d)T_{AB} - \Lambda_{(4+d)}(4+d)g_{AB}.$$ (6.107)

Here we have introduced the gravitational constant in $(4+d)$-dimensions

$$\kappa^2_{(4+d)} = \frac{8\pi}{M^2_{(4+d)}}.$$ (6.108)

If the length scale of the extra dimensions is $L$, then on scales $r < L$ we would have a $(4+d)$-dimensional potential, while in the limit of $r > L$ the potential would behave as a 4D potential, $V \simeq L^{-d}r^{-1}$. This means that the usual Planck scale becomes an effective coupling constant, describing gravity on scales much larger than the extra dimensions

$$M_P \simeq M_{(4+d)}L^d.$$ (6.109)

If the extra dimensional volume is Planck scale, i.e. $L \simeq 1/M_P$, then $M_{(4+d)} \simeq M_P$. But if the extra fundamental scale $M_{(4+d)}$ can be much less than the effective scale $M_P \simeq 10^{19}$ GeV. In this case, the weakness of gravity is due to the fact that it spreads into extra dimensions and only part is felt in 4 dimensions.

In the case of infinitely large extra dimensions, gravity is induced on 4D submanifolds. In this way, at short distances the induced effects dominate, generating an effective 4D Einsteinian gravity consistent with Newton’s experiments. At large distances, the extra dimensional gravity becomes more and more important. This modifies, at large distances, the effective 4D gravity.

Suppose, we have only one extra dimension. We then consider a 3D–domain wall (also called brane) with vacuum energy $\lambda$, where all the matter fields live. The action is then

$$S = \frac{1}{\kappa^5_5} \int d^5x \sqrt{-\bar{g}} \left[ \bar{R} - 2\Lambda_{(5)} \right] + \int d^4x \sqrt{-\bar{g}} \left[ \mathcal{L}_m - 2\lambda \right].$$ (6.110)

$\kappa$: 5D Planck mass; $\bar{g}_{AB}$: 5D metric, and the rest is 5D Ricci scalar, 5D cosmological constant and $g_{\mu\nu}$ is the 4D metric. The total spacetime is called bulk, the 4D brane. The 5D cosmological constant $\Lambda_{(5)}$ is very important, such a term will induce a positive pressure.

there is an underlying theory of which all string theories are only different aspects. This was called M-theory. The M might stand for Mother of all theories or Mystery, because the planet we call M-theory is still largely unexplored.

There is still a third possibility for the M in M-theory. One of the islands that was found on the M-theory planet corresponds to a theory that lives not in 10 but in 11 dimensions. This seems to be telling us that M-theory should be viewed as an 11 dimensional theory that looks 10 dimensional at some points in its space of parameters. Such a theory could have as a fundamental object a Membrane, as opposed to a string. Like a drinking straw seen at a distance, the membranes would look like strings when we curl the 11th dimension into a small circle.
6.6.1 Braneworld Holography

Plato, the great Greek philosopher, wrote a series of ‘Dialogues’ which summarized many of the things which he had learned from his teacher, who was the philosopher Socrates. One of the most famous of these Dialogues is the ‘Allegory of the Cave’. In this allegory, people are chained in a cave so that they can only see the shadows which are cast on the walls of the cave by a fire. To these people, the shadows represent the totality of their existence – it is impossible for them to imagine a reality which consists of anything other than the fuzzy shadows on the wall. However, some prisoners may escape from the cave; they may go out into the light of the Sun and behold true reality. When they try to go back into the cave and tell the other captives the truth, they are mocked as madmen.

Of course, to Plato this story was just meant to symbolize mankind’s struggle to reach enlightenment and understanding through reasoning and open-mindedness. We are all initially prisoners and the tangible world is our cave. Just as some prisoners may escape out into the Sun, so may some people amass knowledge and ascend into the light of true reality. What is equally interesting is the literal interpretation of Plato’s tale: The idea that reality could be represented completely as ‘shadows’ on the walls.

The Holographic Principle and Modern Physics and CFT

In 1993 the famous Dutch theoretical physicist G. ’t Hooft put forward a bold proposal which is reminiscent of Plato’s Allegory of the Cave. This proposal, which is known as the Holographic Principle, consists of two basic assertions:

- The first assertion of the Holographic Principle is that all of the information contained in some region of space can be represented as a ‘Hologram’ – a theory which ‘lives’ on the boundary of that region. For example, if the region of space in question is the our Tearoom, then the holographic principle asserts that all of the physics which takes place in the Tearoom can be represented by a theory which is defined on the walls of the Tearoom.

- The second assertion of the Holographic Principle is that the theory on the boundary of the region of space in question should contain at most one degree of freedom per Planck area. A Planck area is the area enclosed by a little square which has side length equal to the Planck length, a basic unit of length which is usually denoted \( L_P \approx 10^{-33} \text{ cm} \). Remember that the entropy of a black hole is just given by the number of Planck cells of the horizon (so-called Hawking entropy).

To many people, the Holographic Principle seems strange and counterintuitive: How could all of the physics which takes place in a given room be equivalent to some physics defined on the walls of the room? Could all of the information contained in your body actually be represented by your ‘shadow’?

In fact, the way in which the Holographic Principle appears in M–theory is much more subtle. In M–theory we are the shadows on the wall. The ‘room’ is some larger, five–dimensional spacetime and our four–dimensional world is just the boundary of this larger space. If we try to move away from the wall, we are moving into an extra dimension of space – a fifth dimension. In fact, people have recently been trying to think of ways in which we might actually experimentally ‘probe’ this fifth dimension. At the heart of many of these exciting ideas is a version of the Holographic Principle known as the adS/CFT correspondence.
The **AdS/CFT correspondence** is a type of duality, which states that two apparently distinct physical theories are actually equivalent. On one side of this duality is the physics of gravity in a spacetime known as anti–de Sitter space (AdS). Five–dimensional anti-de Sitter space has a boundary which is four–dimensional, and in a certain limit looks like flat spacetime with one time and three space directions. The AdS/CFT correspondence states that the physics of gravity in five–dimensional anti-de Sitter space, is equivalent to a certain supersymmetric Yang-Mills theory which is defined on the boundary of AdS. This Yang-Mills theory is thus a ‘hologram’ of the physics which is happening in five dimensions. The Yang-Mills theory has gauge group \( SU(N) \), where \( N \) is very large, and it is said to be ‘supersymmetric’ because it has a symmetry which allows you to exchange bosons and fermions. The hope is that this theory will eventually teach us something about QCD (quantum chromodynamics), which is a gauge theory with gauge group \( SU_c(3) \). QCD describes interactions between quarks. However, QCD has much less symmetry than the theory defined on the boundary of AdS; for example, QCD has no supersymmetry.

We have Einstein gravity with a negative cosmological constant in the bulk. This is thought to be dual to a field theory on the brane that is cut–off in the UV. We however do not know what this field theory actually is. To study braneworld holography we require two things: a **FRW brane** and a **Black Hole in the bulk**.

The intuition for this goes as follows: the bulk Black Hole emits Hawking radiation that heats the brane to a finite temperature. The braneworld should be hot and have a non–zero energy density and pressure, i.e. the field theory behaves like radiation.

### 6.6.2 Minkowski space as Empty Brane

As an example we consider an **empty brane**, where \( \mathcal{L}_m = 0 \). We want to find a warped solution which has as a slice a Minkowskian spacetime. For this we start with the metric

\[
d s^2 = d y^2 + \exp(-2\sigma(y)) \gamma_{\mu\nu}(x^\alpha) \, d x^\mu \, d x^\nu. \tag{6.111}
\]

\( y \) is the extra coordinate. In order to solve \( \delta S = 0 \), we consider a spacetime with boundary at \( y = 0 \), i.e. the 4 metric varies only on the boundary, but we keep it fixed in the bulk. This leads to a variational principle for the scalar field \( \sigma(y) \) in the bulk and a variational problem for the metric on the brane. For \( \gamma = \eta \), we can decompose the Ricci–scalar (see Appendix)

\[
\sqrt{-\bar{g}} \bar{R} = 12 \exp(-4\sigma) (\sigma')^2 + 2 (K \sqrt{-g})' + \exp(-2\sigma) \bar{R} \tag{6.112}
\]

where \( \bar{R} \) is the Ricci associated with \( \gamma_{\mu\nu} \), and \( K = \frac{1}{2} g^{\mu\nu} L_n \gamma_{\mu\nu} \) is the trace of the extrinsic curvature orthogonal to the brane. This leads to the variational problem

\[
\int d^5 x \sqrt{-\bar{g}} \bar{R} = 12 \int d^5 x \left[ \exp(-4\sigma) (\sigma')^2 + \exp(-2\sigma) \bar{R} \right] + 2 \int d^4 x \sqrt{-g} K, \tag{6.113}
\]

where the boundary is at \( y = 0 \). Since the degree of freedom of the 4D Ricci and that of the scalar field \( \sigma \) are independent, we may set \( R = 0 \). Since \( \delta \sigma = 0 \) on the boundary, we have

\[
12 \delta \int d^5 x \exp(-4\sigma) (\sigma')^2 = -12 \int d^5 x \left[ 2\sigma'' - (4\sigma')^2 \right] \exp(-4\sigma) \delta \sigma. \tag{6.114}
\]

The variation of the 5D cosmological constant is

\[
-2 \delta \int d^5 x \sqrt{-\bar{g}} \bar{\Lambda} = 12 \int d^5 x \frac{2}{3} \bar{\Lambda} \exp(-4\sigma) \delta \sigma. \tag{6.115}
\]
this leads to the condition
\[ \sigma'' - 2(\sigma')^2 = \Lambda / 3. \] (6.116)

This equation has two solutions
\[ \sigma' = \pm \sqrt{-\frac{\Lambda}{6}} \] (6.117)
\[ \sigma' = \pm \left( 1 + \tanh \left( 2\sqrt{-\frac{\Lambda}{6}} y + C \right) \right) \] (6.118)

Only the first solution is the physical one
\[ \sigma' = \sqrt{-\frac{\Lambda(5)}{6}} \operatorname{sign}(y). \] (6.119)

As a result we find then that the bulk, into which Minkowski spacetime is embedded, is
the 5D anti–de Sitter space (AdS)
\[ ds^2 = dy^2 + \exp \left( -2|y|/l \right) \eta_{\mu \nu} dx^\mu dx^\nu. \] (6.120)

where the scale \( l = 1/\sqrt{-\Lambda(5)/6} \) is given by the cosmological constant of the bulk. This is
called the Randall–Sundrum model found in 1999 [21]. The exponential warp factor reflects
the confining role of the bulk cosmological constant.

### 6.6.3 Branes including Matter

The above concept can now be generalized to include matter. For this one requires that the bulk
space is a solution of Einstein’s equations. A brane is given is a slice in the 5D space with
normal \( n^a \) (see Appendix A). This defines the metric induced on the brane
\[ h_{ab} = g_{ab} - n_a n_b. \] (6.121)

The Gauss equation for the curvature (for slicing of manifolds, see Appendix A)
\[ (n-1) R^a_{\ bcd} = (n) R^{e}_{\ fgh} h^a_e h^b_f h^c_g h^d_h + K^a_e K^c_g - K^a_c K^g_h, \] (6.122)

where
\[ K_{ab} \equiv \frac{1}{2} L_n g_{ab} \] (6.123)
is the exterior curvature of the brane. The exterior curvature follows from the Codazzi
equation
\[ D_b (K^b_a - h^b_a K) = (n) R_{cde} n^c h^d_a, \] (6.124)

where \( D_a \) is the covariant derivative made out of \( h_{ab} \). \( K = K^a_a \) is the trace of the exterior
curvature. From here, we also get the relations
\[ -2 (n) G_{ab} n^a n^b = (n-1) R - K^2 + K_{ab} K^{ab}. \] (6.125)
By means of Gauss’ equation we can build the 4D Einstein tensor

\[(n-1)G_{ab} = (n)G_{cd}h_a^c h_b^d + (n)R_{cd} n^c n^d h_{ab} \]

\[+ KK_{ab} - K^c_a K_{bc} - \frac{1}{2} h_{ab}(K^2 - K^{cd} K_{cd}) - E_{ab}, \]  
(6.126)

with

\[E_{ab} = (n)\bar{R}^c_{df} n^c n^f h_a^d h_b^g. \]  
(6.127)

The bulk space is now required to be a solution of Einstein’s equations in 5D

\[(n)G_{ab} = -\Lambda_n g_{ab} + 8\pi G_n \bar{T}_{ab}, \]  
(6.128)

where \(\bar{T}_{ab}\) is the energy–momentum tensor of bulk fields and \(\Lambda_n\) is the cosmological constant of the bulk with Newtonian constant \(G_n\). The braneworld is a 4D hypersurface which can be described by the condition \(y = 0\), where \(y\) is the extra dimensional coordinate. We can now express the Riemann tensor in terms of the Weyl and Ricci tensors

\[R_{abcd} = (n)C_{abcd} + \frac{1}{n-2} \left[ (n)R_{ac}g_{bd} - (n)R_{ad}g_{bc} + (n)R_{bd}g_{ac} - (n)R_{bc}g_{ad} \right] - \frac{1}{(n-1)(n-2)} (n)R \left[ g_{ac}g_{bd} - g_{ad}g_{bc} \right]. \]  
(6.129)

Inserting (6.128) and (6.129) into (6.126) we find Einstein’s equations for the brain

\[(n-1)G_{ab} = -\Lambda_n \frac{n-3}{n-1} h_{ab} - E_{ab} + KK_{ab} - \frac{1}{2} h_{ab}[K^2 - K^{cd} K_{cd}], \]  
(6.130)

where

\[E_{ab} = C_{pqrs} n^p n^q h_a^r h_b^s - 8\pi G_n \frac{n-3}{n-2} \left[ h_a^c h_b^d + n^c n^d h_{ab} - \frac{1}{n-1} g^{cd} h_{ab} \right] \bar{T}_{cd}. \]  
(6.131)

This is often described as the electric part of the Weyl tensor, if there are no extra bulk fields, i.e. \(\bar{T}_{ab} = 0\).

The braneworld is given by the two junction conditions (Israel)

\[[h_{ab}] = 0 \]  
(6.132)

as well as

\[[K_{ab}] = -\kappa^2 \left( S_{ab} - \frac{1}{n-2} h_{ab} S \right), \]  
(6.133)

where the energy–momentum tensor for the brane is given by

\[S_{ab} = \tau_{ab} - \sigma h_{ab} \]  
(6.134)

with \(\tau_{ab} n^b = 0\), i.e. there is no energy flow from the bulk. Here we use the definition

\[[X] = \lim_{y \to +0} X - \lim_{y \to -0} X = X^+ - X^- . \]  
(6.135)
6.6 BraneWorld Cosmology – Beyond Dimension Four

\( \sigma \) is the brane tension and \( \tau_{ab} \) includes all additional matter (perfect fluids e.g.).

For the following we assume \( Z_2 \)-symmetry across the brane, so we have \( [K_{ab}] = 2K_{ab} \). Using the above Israel junction condition, we can replace the extrinsic curvature terms in the Einstein tensor of the brainworld with terms involving \( \sigma \) and \( \tau_{ab} \)

\[
(n-1)G_{ab} = -\Lambda_{n-1} h_{ab} + 8\pi G_{n-1} \tau_{ab} + (4\pi G_n)^2 \Pi_{ab} - E_{ab},
\]

(6.136)

where we have defined the following quantities

\[
\Lambda_{n-1} = \frac{(n-2)(n-3)}{2} \left[ \sigma_n^2 + \frac{2\Lambda_n}{(n-1)(n-2)} \right],
\]

(6.137)

\[
\sigma_n = \frac{4\pi G_n \sigma}{n-2},
\]

(6.138)

\[
G_{n-1} = \frac{n-3}{2} G_n \sigma_n,
\]

(6.139)

\[
\Pi_{ab} = -\tau_c \tau_{bc} + \frac{1}{n-2} \tau \tau_{ab} + \frac{1}{2} \tau^{cd} \tau_{cd} h_{ab} - \frac{1}{2n-4} \tau^2 h_{ab}.
\]

(6.140)

Effective 4D gravity has therefore a source term which is quadratic in the energy–momentum tensor and one part which includes curvature effects from the embedding. \( \lambda_{n-1} \) is the braneworld cosmological constant and \( G_{n-1} \) Newton’s constant. Another important term is the Weyl tensor \( E_{ab} \). It contains information about the bulk, but is constrained to the matter on the brane.

A FRW Brane

We assume that the bulk spacetime has negative cosmological constant and has no additional fields, i.e.

\[
\Lambda_n = \frac{(n-1)(n-2)}{2} k_n^2, \quad T_{ab} = 0.
\]

(6.141)

The cosmological constant on the brane is then

\[
\Lambda_{n-1} = \frac{(n-1)(n-2)}{2} [\sigma_n^2 - k_n^2].
\]

(6.142)

Branes with \( \sigma_n = k_n \) are called critical branes. Now we assume a braneworld metric of the FRW form

\[
ds_{n-1}^2 = -d\tau^2 + a^2(\tau) d\sigma_n^2.
\]

(6.143)

The brane contains a perfect fluid

\[
\tau_{ab} = \rho \tau_a \tau_b + P(\tau_a \tau_b + h_{ab}),
\]

(6.144)

where \( \tau^a \) are the components of \( \partial / \partial \tau \). Further \( E_{ab} = 0 \), which corresponds to a pure anti–de Sitter space in the bulk. The FRW equations now follow from the braneworld Einstein’s equations. Defining \( H \equiv \dot{a} / a \) as the Hubble parameter, we find

\[
H^2 = \frac{\sigma_n^2 - k_n^2}{a^2} + \frac{16\pi G_{n-1}}{(n-2)(n-3)} \rho + \left( \frac{4\pi G_n}{n-2} \right)^2 \rho^2
\]

(6.145)

\[
\dot{H} = \frac{k}{a^2} - \frac{8\pi G_{n-1}}{n-3} \left( \rho + P \right) - (n-2) \left( \frac{4\pi G_n}{n-2} \right)^2 \rho(\rho + P).
\]

(6.146)
These are not standard Friedmann equations, since they contain quadratic terms in $\rho$. Braneworld cosmology is therefore different to standard cosmology. This unconventional behaviour was first discovered in 5 dimensions by Binetruy et al. in 2000. However, for large scale factors we recover the standard cosmology, since the nonlinear density terms drop out.

### 6.6.4 Bulk Based BraneWorld Cosmology

In the bulk based approach we consider a dynamic brane in a static bulk. But we only consider FRW branes. When we have a static bulk, we immediately think on Birkhoff’s theorem:

**if the geometry of a given region of spacetime is spherically symmetric and a solution to the vacuum Einstein equations, then it is necessarily a piece of the Schwarzschild geometry.**

A generalized version in 5D was proven by Bowcock et al. Actually it can be proven in $n$ dimensions. The bulk metric can then be written with the following ansatz (similar to the 4D Schwarzschild ansatz)

$$ds^2 = -h(y) \, dt^2 + \frac{dy^2}{h(y)} + y^2 \, d\sigma_k ,$$

(6.147)

where $d\sigma_k$ is the line element of a $(n - 2)$-dimensional space of constant curvature. The solution has the form

$$h(y) = k - \frac{\omega_d M}{y^{n-3}} + \frac{y^2}{l^2},$$

(6.148)

where

$$\omega_d = \frac{16\pi G}{(n-2)Vol(M^{n-2})}$$

(6.149)

and $Vol(M^{n-2}) = \int d^{n-2} x \sqrt{h}$. The parameter $\omega_d$ is inserted for convenience such that the parameter $M$ has the dimension of inverse length. $l$ is related to the cosmological constant in the bulk, $l^2 = (n-1)(n-2)/2\Lambda_n$. With this form one can check that the bulk is an Einstein space with negative cosmological constant

$$R_{AB} = -\frac{n-1}{l^2} g_{AB},$$

(6.150)

provided the horizon is an Einstein space of the form $R_{ij} = (n-3)k h_{ij}$.

A Black Hole solution is achieved provided the polynomial $y^{n-3}l^2 h$ has a simple positive root $y_+$, such that $h(y) > 0$ for all $y > y_+$. For $k = 1$ and spherical topology, these solutions have been discussed by Carter (1968) and Brown et al. (1994). For $k = 0$, one can check that for $M > 0$ there is always a simple positive root of $y^{n-3}l^2 h$ given by

$$y_+ = (\omega_d M l^2)^{1/(n-1)}.$$

(6.151)
6.6.5 Quantum BraneWorlds

The classical Einstein theory can be considered as the weak field and low energy limit of some unknown quantum gravity model. As pointed out by Boulware and Deser (1985), among higher order curvature corrections to the GR action, the quadratic term is especially important as it is the leading one. Such a term is naturally generated in heterotic string theory, and it has been used in cosmology to address the cosmological constant problem [14].

In the above treatment of the cosmology of the BraneWorld, the domain–wall (brane) is described as a 4D world–volume slice of a 5D SpaceTime, which is of constant curvature (Binetruy et al. 2000). This corresponds to a 3–surface of a RW world–volume moving in the background of a 5D AdS–Schwarzschild black hole. Quantum gravity adds terms quadratic in the curvature to the classical action, since Einstein’s theory is only the weak limit of a more complex theory of gravity. Of particular importance is the Gauss–Bonnet action, where the resulting field equations contain no more than second order derivatives of the metric and it has been proven to be free of ghosts when expanding about flat space.

The 5D bulk action including a Gauss–Bonnet BraneWorld scenario is given by

$$S_{BW} = \frac{1}{2\kappa_5^2} \int_M d^5x \sqrt{-g} \left[ R - 2\Lambda_5 + \alpha_{GB} \left( R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd} \right) \right] + S_{\partial M} + S_{\text{mat}}. \quad (6.152)$$

$$\alpha_{GB} > 0 \text{ represents the Gauss–Bonnet coupling with dimension } (\text{length})^2, \Lambda_5 < 0 \text{ is the bulk cosmological constant, } \kappa_5^2 = 8\pi/5M_5^3 \text{ the 5D Planck scale. } S_{\partial M} \text{ is a boundary term required to cancel normal derivatives of the metric that arise when varying the action with respect to the metric. } S_{\text{mat}} \text{ describes matter on the brane}$$

$$S_{\text{mat}} = \int_{\partial M} d^4x \sqrt{-h} L_{\text{mat}}, \quad (6.153)$$

where \( h \) is the metric induced on the brane.

**AdS–Schwarzschild Black Hole**

For the bulk metric ignoring matter we can assume the following spherically symmetric form

$$ds^2 = -\exp(2\nu) dt^2 + \exp(2\lambda) dy^2 + y^2 h_{ij} dx^i dx^j, \quad (6.154)$$

where \( \nu \) and \( \lambda \) are only functions of \( y \) and \( h_{ij} dx^i dx^j \) represents the line element of a \((n-2)\)-dimensional hypersurface with constant curvature \((n-2)(n-3)k \) and volume \( \Sigma_k \). The solution has the form [5]

$$h(y) = \exp(2\nu) = \exp(-2\lambda) = k + \frac{y^2}{2\tilde{\alpha}} \left( 1 \mp \sqrt{1 + \frac{64\pi G\tilde{\alpha}M}{(n-2)\Sigma_k y^{n-1}} - \frac{4\tilde{\alpha}}{\tilde{l}^2}} \right), \quad (6.155)$$

where \( M \) is the gravitational mass of the solution. The solution with \( k = 1 \) was first found by Boulware and Deser [4]. \( \tilde{\alpha} = (n-3)(n-4)\alpha_{GB} \) and \( \tilde{l} \) is related to the bulk cosmological constant \( \Lambda_5 = -(n-1)(n-2)/\tilde{l}^2 \).

\(^2\)The gravity theory with a Gauss–Bonnet term was originally proposed by Lanczos (1938), and it has been rediscovered by Lovelock in 1973.
This solution represents a topological black hole where the event horizon can be a hypersurface with positive, zero or negative constant curvature (AdS–Schwarzschild black hole).

**Robertson–Walker as a 3–Wall of constant Curvature in AdS–Schwarzschild**

We now consider a 3–wall of constant spatial curvature in the background of the above black hole solution of the Gauss–Bonnet 5D SpaceTime. Its line element can then be written in the form

\[ ds^2 = -f(y) \, dt^2 + \frac{dy^2}{h(y)} + y^2 \, d\sigma_3^2 \]

(6.156)

\[ d\sigma_3^2 = \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \]

(6.157)

\[ h(y) = k + \frac{y^2}{4\alpha_{GB}} \left( 1 \pm \sqrt{1 + \frac{4\alpha_{GB}\Lambda_5}{3} + \frac{8\alpha_{GB}M_H}{y^4}} \right). \]

(6.158)

We now transform from the black hole coordinates to the ones built around the trajectory of the wall [19]

\[ ds^2 = -\psi^2(\tau, w) \phi(\tau, w) \, d\tau^2 + a^2(\tau) \phi(\tau, w) \, d\sigma_3^2 + dw^2. \]

(6.159)

The coordinate \( w \) is the proper time of the spacelike geodesics that cross the trajectory vertically. The induced metric at \( w = 0 \) is Robertson–Walker with scale factor \( a(\tau) \) and Hubble parameter \( H = \dot{a}(\tau)/a(\tau) \).

The derivation of the Friedmann equation is quite lengthy (see e.g. [?] using the Cartan formalism) and it can be written as

\[ H^2 = \frac{c_{\pm} + c_{-} - \frac{2}{8\alpha_{GB}}}{c_{\pm}} , \]

(6.160)

with

\[ c_{\pm} = \left\{ \sqrt{\left( 1 + \frac{4\alpha_{GB}\Lambda_5}{3} \right)^{3/2} + \frac{\alpha_{GB}\kappa_5^2\sigma^2}{2} \pm \sqrt{\frac{\alpha_{GB}}{2}\kappa_5^2\sigma}} \right\}^{2/3}. \]

(6.161)

\( \sigma \) is the total energy density of matter sources which satisfies the conservation equation

\[ \dot{\sigma} + 3H(\sigma + P) = 0. \]

(6.162)

With the transformation

\[ \sigma = \sqrt{\frac{2b}{\alpha_{GB}\kappa_5^2}} \sinh(x), \]

(6.163)

where the constant \( b \) is defined as

\[ b \equiv (1 + 4\alpha_{GB}\Lambda_5/3)^{3/2}, \]

(6.164)

we arrive at

\[ c_{\pm} = b^{1/3} \exp(\pm 2x/3). \]

(6.165)
Therefore, the Friedmann equation simplifies, but is highly non–linear in the energy density $\sigma$

$$H^2 = \frac{1}{4\alpha_{GB}} \left[ b^{1/3} \cosh(2x/3) - 1 \right]. \quad (6.166)$$

We can expand this to second order in the matter field

$$H^2 = \frac{\kappa^2_4}{36\alpha_{GB}} \sigma^2 + \frac{1}{4\alpha_{GB}} (b^{1/3} - 1). \quad (6.167)$$

Energy density on the brane consists of two contributions, $\sigma = \rho + \lambda$ with the brane tension $\lambda > 0$, resulting in the typical quadratic Friedmann equation of BraneWorld cosmology

$$H^2 = \frac{\kappa^2_4}{3} \rho \left(1 + \frac{\rho}{2\lambda}\right) + \frac{\Lambda_4}{3} \quad (6.168)$$

with the 4D cosmological constant

$$\Lambda_4 = \frac{3}{4\alpha_{GB}} (b^{1/3} - 1) + \frac{\kappa^4_5}{12b^{2/3}} \lambda^2 \quad (6.169)$$

and the 4D Newtonian coupling constant

$$\kappa^4_2 = \frac{8\pi}{M_P^2} = \frac{\kappa^4_2}{6b^{2/3}}. \quad (6.170)$$

For $\Lambda_4 = 0$, the brane tension is given by

$$\lambda = \frac{3}{2} (1 - b^{1/3}) \frac{1}{\alpha_{GB}\kappa^4_4}. \quad (6.171)$$

### 6.6.6 What is the Ekpyrotic Universe?

Because the Big bang model, with no amendments, would tend to produce a universe that is highly inhomogeneous, with a warped and curved space, and no natural mechanism for making stars, galaxies and larger scale structures in the universe. Cosmologists have been trying to correct these deficiencies by amending the early history of the universe - within the first billionth billionth billionths of second or less. One proposal is the “inflationary theory” of the universe, which proposes that the universe began hot and dense, and underwent a period of hyperexpansion. The ekpyrotic model is a new alternative, which is, in many ways, a more radical departure from the Big Bang concept.

The model is based on the idea that our hot big bang universe was created from the collision of two three-dimensional worlds moving along a hidden, extra dimension. The two three-dimensional worlds collide and “stick,” the kinetic energy in the collision is converted the quarks, electrons, photons, etc., that are confined to move along three dimensions. The resulting temperature is finite, so the hot big bang phase begins without a singularity. The universe is homogeneous because the collision and initiation of the big bang phase occurs nearly

---

3The term *ekpyrosis* means “conflagration” in Greek, and refers to an ancient Stoic cosmological model. According to the model, the universe is created in a sudden burst of fire, not unlike the collision between three–dimensional worlds in our model. The current universe evolves from the initial fire.
simultaneously everywhere. The energetically preferred geometry for the two worlds is flat, so their collision produces a flat big bang universe. According to Einstein’s equations, this means that the total energy density of the Universe is equal to the critical density. Massive magnetic monopoles, which are overabundantly produced in the standard big bang theory, are not produced at all in this scenario because the temperature after collision is far too small to produce any of these massive particles.

Quantum effects cause the incoming three-dimensional world to ripple along the extra-dimension prior to collision so that the collision occurs in some places at slightly different times than others. By the time the collision is complete, the rippling leads to small variations in temperature, which seed temperature fluctuations in the microwave background and the formation of galaxies. We have shown that the spectrum of energy density fluctuations is scale-invariant (the same amplitude on all scales). The production of a scale-invariant spectrum from hyperexpansion was one of the great triumphs of inflationary theory, and here we have repeated the feat using completely different physics.

The building blocks of the ekpyrotic theory are derived from superstring theory. Superstring theory requires extra dimensions for mathematical consistency. In most formulations, 10 dimensions are required. In the mid-1990’s, Petr Horava (Rutgers) and Ed Witten (IAS, Princeton) argued that, under certain conditions, an additional dimension opens up over a finite interval. Six dimensions are presumed to be curled up in a microscopic ball, called a Calabi-Yau manifold. The ball is too small to be noticed in everyday experience, and so our universe appears to be a four-dimensional (three space dimensions and one time dimension) surface embedded in a five-dimensional space-time. This five-dimensional theory, called heterotic M-theory, was formulated by Andre Lukas (Sussex), Ovrut and Dan Waldram (Queen Mary Westerfield College). According to Horava-Witten and heterotic M-theory, particles are constrained to move on one of the three-dimensional boundaries on either side of the extra dimensional interval. Our visible universe would be one of these boundaries; the other boundary and the intervening space would be hidden because particles and light cannot not travel across the intervening space. Only gravity is able to couple matter on one boundary to the other. In addition, there can exist other three-dimensional hypersurfaces in the interval, which lie parallel to the outer boundaries and which can carry energy. These intervening planes are called “branes,” short for membranes. The collision that ignites the hot big bang phase of the ekpyrotic model occurs when a three-dimensional brane is attracted to and collides into the boundary corresponding to our visible universe.

### 6.7 String Theory vs Loop Quantum Gravity

The problem of describing the quantum regime of the gravitational field is still open. There are tentative theories, and competing research directions. The two largest research programs are string theory and loop quantum gravity. But several other directions are being explored, such as twistor theory, noncommutative geometry, simplicial quantum gravity, Euclidean quantum gravity, the Null Surface Formulation and others. String theory and loop quantum gravity differ not only because they explore distinct physical hypotheses, but also because they are expressions of two separate communities of scientists, scientists who have sharply distinct prejudices, and who view the problem of quantum gravity in surprisingly different manners.

High energy physics has obtained spectacular successes during this century, culminating with the (far from linear) establishment of quantum field theory as the general form of dy-
nematics and with the comprehensive success of the Standard Model. Thanks to this success, now a few decades old, physics is in a position in which it has very rarely been: There are no experimental results that clearly challenge, or clearly escape, the present fundamental theory of the world. The theory we have encompasses virtually everything - except gravitational phenomena. From the point of view of a particle physicist, gravity is then simply the last and weakest of the interactions. It is natural to try to understand its quantum properties using the strategy that has been so successful for the rest of microphysics, or variants of this strategy.

The search for a conventional quantum field theory capable of embracing gravity has spanned several decades and, through an adventurous sequence of twists, moments of excitement and disappointments, has lead to string theory. The foundations of string theory are not yet well understood; and it is not yet entirely clear how a supersymmetric theory in 10 or 11 dimensions can be concretely used for deriving comprehensive univocal predictions about our world. But string theory may claim extremely remarkable theoretical successes and is today the leading and most widely investigated candidate theory of quantum gravity.

In string theory, gravity is just one of the excitations of a string (or other extended object) living over some background metric space. The existence of such background metric space, over which the theory is defined, is needed for the formulation and for the interpretation of the theory. This is the case not only in perturbative string theory, in the recent attempts at a non-perturbative definition of the theory, such as M–theory. Thus, for a physicist with a high energy background, the problem of quantum gravity is now reduced to an aspect of the problem of understanding: What is the mysterious nonperturbative theory that has perturbative string theory as its perturbation expansion? And how does one extract information on Planck scale physics from it?

For a relativist, on the other hand, the idea of a fundamental description of gravity in terms of physical excitations over a background metric space sounds physically very wrong. The key lesson learned from general relativity is that there is no background metric over which physics happens (unless, of course, in approximations). The world is more complicated than that. Indeed, for a relativist, general relativity is much more than the field theory of a particular force. Rather, it is the discovery that certain classical notions about space and time are inadequate at the fundamental level; they require modifications which are possibly as basic as the ones that quantum mechanics introduced. One of these inadequate notions is precisely the notion of a background metric space, (flat or curved), over which physics happens. This profound conceptual shift has led to the understanding of relativistic gravity, to the discovery of black holes, to relativistic astrophysics and to modern cosmology.

From Newton to the beginning of this century, physics has had a solid foundation in a small number of key notions such as space, time, causality and matter. In spite of substantial evolution, these notions remained rather stable and self-consistent. In the first quarter of this century, quantum theory and general relativity have deeply modified this foundation. The two theories have obtained solid success and vast experimental corroboration, and can now be considered established knowledge. Each of the two theories modifies the conceptual foundation of classical physics in an (more or less) internally consistent manner, but we do not have a novel conceptual foundation capable of supporting both theories. This is why we do not yet have a theory capable of predicting what happens in the physical regime in which both theories are relevant, the regime of Planck scale phenomena, $L_P \simeq 10^{-33}$ cm.

Thus, for a relativist, the problem of quantum gravity is the problem of bringing a vast conceptual revolution, begun with quantum mechanics and with general relativity, to a conclusion and to a new synthesis. In this synthesis the notions of space and time need to be
The Inflationary Universe, BraneWorld Cosmology and Quantum Gravity deeply reshaped in order to take into account what we have learned with both our present “fundamental” theories. Unlike perturbative or nonperturbative string theory, loop quantum gravity is formulated without a background spacetime. Loop quantum gravity is thus a genuine attempt to grasp what quantum spacetime is at the fundamental level. Accordingly, the notion of spacetime that emerges from the theory is profoundly different from the one on which conventional quantum field theory or string theory is based [22].

The main merits of string theory are that it provides a superbly elegant unification of known fundamental physics, and that it has a well defined perturbation expansion, finite order by order. Its main incompletenesses are that its non-perturbative regime is poorly understood, and that we do not have a background-independent formulation of the theory. In a sense, we do not really know what the theory we are talking about is. Because of this poor understanding of the non-perturbative regime of the theory, Planck scale physics and genuine quantum gravitational phenomena are not easily controlled: Except for a few computations, there has not been much Planck scale physics derived from string theory so far. There are, however, two sets of remarkable physical results. The first is given by some very high energy scattering amplitudes that have been computed. An intriguing aspect of these results is that they indirectly suggest that geometry below the Planck scale cannot be probed -and thus in a sense does not exist- in string theory. The second physical achievement of string theory (which followed the d-branes revolution) is the recent derivation of the Bekenstein-Hawking black hole entropy formula for certain kinds of black holes.

The main merit of loop quantum gravity, on the other hand, is that it provides a well-defined and mathematically rigorous formulation of a background-independent, non-perturbative generally covariant quantum field theory. The theory provides a physical picture and quantitative predictions of the world at the Planck scale. The main incompleteness of the theory is regarding the dynamics, formulated in several variants. So far, the theory has lead to two main sets of physical results. The first is the derivation of the (Planck scale) eigenvalues of geometrical quantities such as areas and volumes. The second is the derivation of black hole entropy for “normal” black holes (but only up to the precise numerical factor).

The main idea beyond loop quantum gravity is to take general relativity seriously. We have learned with general relativity that the spacetime metric and the gravitational field are the same physical entity. Thus, a quantum theory of the gravitational field is a quantum theory of the spacetime metric as well. It follows that quantum gravity cannot be formulated as a quantum field theory over a metric manifold, because there is no (classical) metric manifold whatsoever in a regime in which gravity (and therefore the metric) is a quantum variable.

One could conventionally split the spacetime metric into two terms: one to be considered a background, which gives a metric structure to spacetime; the other to be treated as a fluctuating quantum field. This, indeed, is the procedure on which old perturbative quantum gravity, perturbative strings, as well as current non-perturbative string theories (M-theory), are based. In following this path, one assumes, for instance, that the causal structure of spacetime is determined by the underlying background metric alone, and not by the full metric. Contrary to this, in loop quantum gravity we assume that the identification between the gravitational field and the metric-causal structure of spacetime holds, and must be taken into account, in the quantum regime as well. Thus, no split of the metric is made, and there is no background metric on spacetime.

We can still describe spacetime as a (differentiable) manifold (a space without metric structure), over which quantum fields are defined. A classical metric structure will then be defined by expectation values of the gravitational field operator. Thus, the problem of quan-
6.8 Baryogenesis via Leptogenesis

In hybrid inflationary models, it is preferable to produce an initial lepton asymmetry which is then partly turned into baryon asymmetry. Thereby, the inflaton decays into right-handed neutrino superfields. Their subsequent decay into lepton (antilepton) $L$ ($\bar{L}$) can produce a lepton asymmetry.

In a pure $SU(2)_L$ gauge theory the baryon and lepton number currents carry anomalies

$$\partial_{\mu} J_B^{\mu} = \partial_{\mu} J_L^{\mu} = -n_g \frac{g^2}{16\pi^2} tr(F^*_{\mu\nu}F^{\mu\nu}),$$

(6.172)

where $n_g$ is the number of generations. $F_{\mu\nu}$ is the field tensor for topological instanton solutions which induce tunneling between different states. This tunneling is accompanied by a change of the baryon and lepton number. This implies a non–vanishing baryon number $\frac{n_B}{s}$.

$$\frac{n_B}{s} = \frac{4(1 + 2n_g) n_{B-L}}{22n_g + 13} s.$$

(6.173)

In this respect it is crucial to generate a primordial $n_{B-L}/s$ and not only $n_B/s$, since otherwise the final $n_B/s$ will vanish.

6.9 Conclusions

Inflationary cosmology suggests that the universe in its early stages underwent a period of exponential expansion driven by an almost constant vacuum energy density. This may have happened during the GUT phase transition at which the Higgs field which breaks the GUT gauge symmetry was displaced from vacuum. This filed (the so-called inflaton) could then for some time roll slowly towards the vacuum providing the vacuum energy density. This inflation generates the primordial density perturbations which are necessary for the large scale structure formation and the observed temperature fluctuations of the CMBR. After the end of inflation, the inflaton performs damped oscillations about the vacuum value and eventually decays into light particles reheating thereby the universe.
The early realisations of inflation required unnaturally small coupling constants. This problem can be solved by the so-called hybrid inflationary scenario which uses two real scalar fields instead of one. One of them provides the vacuum energy density for inflation, while the second one is slowly rolling field. Hybrid inflation arises naturally in many SUSY GUTs. In these models, adequate baryogenesis via a primordial leptogenesis occurs consistently with the solar and atmospheric neutrino oscillation data. The primordial lepton asymmetry is turned partly into baryon asymmetry via the electroweak sphaleron effects.

6.10 Exercises

**Simple Inflation:** Consider the potential \( V(\Phi) = \frac{m^2}{2} \Phi^2 \) with \( m \) as the mass of the scalar field. Write the slow–roll equations and derive the values for the slow–roll parameters \( \epsilon \) and \( \eta \). Show that the slow–roll equations are solved by

\[
\Phi(t) = \Phi_i - \frac{m M_P}{\sqrt{12\pi}} t
\]

and

\[
a(t) = a_i \exp \left[ \sqrt{\frac{4\pi}{3}} \frac{m}{M_P} \left( \Phi_i t - \frac{m M_P}{\sqrt{48\pi}} t^2 \right) \right].
\]

Derive the e–fold \( N \).

**Chaotic Inflation:** Show that the expressions given for \( a(t) \) and \( \Phi(t) \) solve the dynamics for chaotic inflation, \( V(\Phi) = \lambda \Phi^4 \).

**FRW BraneWorlds**

Consider Einstein’s equations in the bulk with cosmological constant

\[
G_{AB} + \Lambda_5 g_{AB} = 0,
\]

with the field equations on the brane

\[
M_P^2 G_{AB} + \sigma h_{AB} = \tau_{AB} + M_5^3(K_{AB} - Kg_{AB}).
\]

Here, \( \tau_{AB} \) is the stress–energy on the brane which results from the last term in the above Langrangian. The last term in the braneworld Einstein’s equations makes this theory different from Einstein’s equations in the normal world. The variation of the surface term \( \int_{brane} K \) leads to Israel’s junction conditions. We can use this equation to show that there is a scale whose value determines the validity of General Relativity. For this we define \( l \equiv 2M_P^2/M_5^3 \). Typically we find \( G_{AB} \approx 1/r_1^2 \) and \( K_{AB} \approx 1/r_2^2 \), so that for \( r_1^2 \ll l r_2 \), the last term on the rhs can be neglected, and General Relativity is recovered. In cosmology, we often have \( r_1 \approx r_2 \), and therefore **GR is valid on small scales** \( r \ll l \). Braneworld models can depart from standard FRW models at late times and on large scales.

By contracting the Gauss identity

\[
R^A_{BCD} = R^E_{FGH}h^A_E h^F_B h^G_C h^K_D + K_C K_{BD} - K_B^A K_{CD}
\]

(6.178)
and using the bulk Einstein equations we get the constraints equation

\[ R - 2\Lambda_b + K_{AB}K^{AB} - K^2 = 0. \] (6.179)

This can be written as a scalar equation on the brane

\[ M_6^2(R - 2\Lambda_b) + (M_6^2G_{AB} + \sigma h_{AB} - \tau_{AB})(M_6^2G^{AB} + \sigma h^{AB} - \tau^{AB}) - \frac{1}{3}(M_6^2R - 4\sigma + \tau)^2 = 0, \] (6.180)

where \( \tau = h^{AB}\tau_{AB} \). One method of obtaining solutions is first to solve the constraint equation on the brane together with stress–energy conservation and then integrating the Einstein’s equations in the bulk with given data on the brane. Equation (6.179) describes the evolution on the brane in terms of intrinsic quantities. All homogeneous and isotropic models can be obtained from this equation [7]. All these solutions are embeddable into the 5D Schwarzschild–AdS bulk.

The gravitational equations in the bulk can be integrated by using normal, Gaussian coordinates, written in coordinates \((y, x)\)

\[ ds^2 = dy^2 + h_{\alpha\beta}(y, x)dx^\alpha dx^\beta. \] (6.181)

For any hypersurface \( y = \text{const} \) in the bulk, we have the following embedding equations

\[
\begin{align*}
\frac{\partial K_{\beta}}{\partial y} &= R_{\beta} - KK_{\beta} - \frac{1}{6}\delta_{\beta}^{\alpha}(R + 2\Lambda_b - K_{\mu}K_{\mu} - K^2) \\
&= R_{\beta} - KK_{\beta} - \frac{2}{3}\delta_{\beta}^{\alpha}\Lambda_b \\
\frac{\partial h_{\alpha\beta}}{\partial y} &= 2h_{\alpha\gamma}K_{\gamma}^{\beta}.
\end{align*}
\] (6.182)

These two equations together with the above constraint equation represent the 4+1 splitting of Einstein’s equations in the Gaussian normal coordinates.

**Vacuum Branes:** In the cosmological context, such solutions are achieved in the course of the comological evolution, since matter content decays away. For this we consider solutions for \( \tau_{AB} = 0 \). The constraints equation (6.179) requires

\[
\left( M_6^2 + \frac{2}{3}\sigma M_p^2 \right) R + M_p^4 \left( R_{AB}R^{AB} - \frac{1}{3}R^2 \right) - 4M_6^2\Lambda_{RS} = 0,
\] (6.184)

where

\[ \Lambda_{RS} = \frac{\Lambda_b}{2} + \frac{\sigma^2}{3M_6^2}. \] (6.185)

A possible solution is e.g. the closed de Sitter spacetime

\[ ds^2 = dt^2 - \frac{1}{H^2} \cosh^2(Ht) \left[ d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\phi^2) \right], \] (6.186)

where \(-\infty < t < \infty, 0 \leq \chi, \theta < \pi \) and \( 0 \leq \phi < 2\pi \). The four–dimensional metric has the property that \( R_{\alpha}^{\alpha} = 3H^2h_{\alpha}^{\alpha} \). This was one of the first inflationary model sustained by conformal quantum anomalies [24].
This de Sitter spacetime belongs to the class of spacetimes which satisfy Einstein’s equations with a cosmological constant

$$G_{ab} = \Lambda h_{ab}. \quad (6.187)$$

They satisfy the constraint equation

$$\Lambda = \frac{1}{M_5^2} \left[ \left( \frac{3M_6^6}{2M_5^2} + \sigma \right) \pm \sqrt{\left( \frac{3M_6^6}{2M_5^2} + \sigma \right)^2 - 3M_5^6 \Lambda_{RS}} \right]. \quad (6.188)$$

This equation expresses the resulting cosmological constant on the brane in terms of the coupling constants of the theory. In the frequently discussed special case $M_P = 0$ (Randall–Sundrum model), one obtains $\Lambda_b = \Lambda_{RS}$. The two signs correspond to the two different ways in which the lower-dimensional brane can form a boundary of the higher-dimensional bulk. The condition $\Lambda_{RS} = 0$ is the fine-tuning of Randall and Sundrum.

The above equation is only meaningful when the expression under the square root is non-negative. This leads to the following conclusion: a universe that contains matter and satisfies

$$\frac{3M_6^6 \Lambda_{RS}}{(3M_6^6/2M_5^2 + \sigma)^2} > 1 \quad (6.189)$$

cannot expand forever.

**Cosmological Solutions:** For homogeneous and isotropic branes, the constraint equation (6.179) can be integrated

$$M_4^4 \left( H^2 + \frac{k}{a^2} + \frac{\rho + \sigma}{3M_6^6} \right)^2 = M_6^6 \left( H^2 + \frac{k}{a^2} - \frac{\Lambda_b}{6} - \frac{C}{a^4} \right). \quad (6.190)$$

$C$ is an integration constant, $H = \dot{a}/a$ is the Hubble parameter, and $k = 0, \pm 1$ the curvature of the brane. This equation can be solved with respect to the Hubble parameter

$$H^2 + \frac{k}{a^2} = \frac{\rho + \sigma}{3M_6^6} + \frac{2}{l^2} \left[ 1 \pm \sqrt{1 + l^2 \left( \frac{\rho + \sigma}{3M_5^2} - \frac{\Lambda_b}{6} - \frac{C}{a^4} \right)} \right], \quad (6.191)$$

where the length scale

$$l = \frac{2M_5^2}{M_6^6} \quad (6.192)$$

has been introduced previously. Again the two signs corresponds to two different ways of embedding the brane into the AdS.

The decaying term $C/a^4$ is called the dark radiation (gravitational waves?) and is assumed to be negligibly small in the present universe. Similar to GR cosmology, we may define
density parameters

\[
\Omega_M = \frac{\rho_0}{3M_p^2 H_0^2} \quad (6.193)
\]

\[
\Omega_k = -\frac{k}{\sigma_0^2 H_0^2} \quad (6.194)
\]

\[
\Omega_\sigma = \frac{\sigma_0^2 M_p^2 H_0^2}{3} \quad (6.195)
\]

\[
\Omega_l = \frac{1}{l^2 H_0^2} \quad (6.196)
\]

\[
\Omega_{\Lambda_b} = -\frac{\Lambda_b}{6H_0^2} \quad (6.197)
\]

This equation represents a generalized Friedmann equation which now contains quadratic extensions. The two solutions for the Friedmann equation can be written as

- **BRANE1:**

  \[
  \frac{H^2(z)}{H_0^2} = \Omega_M (1+z)^3 + \Omega_k (1+z)^2 + \Omega_\sigma + 2\Omega_l - 2\sqrt{\Omega_l} \sqrt{\Omega_M (1+z)^3 + \Omega_\sigma + \Omega_l + \Omega_{\Lambda_b}}.
  \]  

- **BRANE2:**

  \[
  \frac{H^2(z)}{H_0^2} = \Omega_M (1+z)^3 + \Omega_k (1+z)^2 + \Omega_\sigma + 2\Omega_l + 2\sqrt{\Omega_l} \sqrt{\Omega_M (1+z)^3 + \Omega_\sigma + \Omega_l + \Omega_{\Lambda_b}}.
  \]  

The first three terms are just the terms of GR. The constraints at present time can be written in the form of

\[
1 - \Omega_k + \Omega_{\Lambda_b} = \left(\sqrt{\Omega_M + \Omega_\sigma + \Omega_l + \Omega_{\Lambda_b}} \pm \sqrt{\Omega_l}\right)^2.
\]  

(6.200)

The theory only makes sense in the limit of \(\Omega_M + \Omega_\sigma + \Omega_l + \Omega_{\Lambda_b} > 0\). Since \(\Omega_l > 0\), we have to distinguish between the case \(\Omega_M + \Omega_\sigma + \Omega_{\Lambda_b} \geq 0\), i.e.

\[
\sqrt{1 - \Omega_k + \Omega_{\Lambda_b}} \mp \sqrt{\Omega_l} = \sqrt{\Omega_l + \Omega_M + \Omega_\sigma + \Omega_{\Lambda_b}}.
\]  

(6.201)

and therefore by taking the square of the above equation, we get

\[
\Omega_M + \Omega_k + \Omega_\sigma \pm 2\sqrt{\Omega_l} \sqrt{1 - \Omega_k + \Omega_{\Lambda_b}} = 1.
\]  

(6.202)

In the opposite case of

\[-\Omega_l < \Omega_M + \Omega_\sigma + \Omega_{\Lambda_b} < 0\]  

(6.203)

we get

\[
\sqrt{1 - \Omega_k + \Omega_{\Lambda_b}} - \sqrt{\Omega_l} = \pm \sqrt{\Omega_l + \Omega_M + \Omega_\sigma + \Omega_{\Lambda_b}}.
\]  

(6.204)
Taking the square of both sides this yields
\[ \Omega_M + \Omega_k + \Omega_\sigma + \sqrt{\Omega_l} \sqrt{1 - \Omega_k + \Omega_\Lambda_b} = 1 . \quad (6.205) \]

To summarize, the constraint equations lead to
\[ \Omega_M + \Omega_k + \Omega_\sigma - 2\sqrt{\Omega_l} \sqrt{1 - \Omega_k + \Omega_\Lambda_b} = 1 , \quad (BRANE1) \quad (6.206) \]
\[ \Omega_M + \Omega_k + \Omega_\sigma + 2\sqrt{\Omega_l} \sqrt{1 - \Omega_k + \Omega_\Lambda_b} = 1 , \quad (BRANE2) . \quad (6.207) \]

Bibliography

A Hypersurface Embeddings

Let us consider the embedding of a hypersurface \( \Sigma \) into an 5–dimensional manifold. The metric \( \gamma \) contains the complete information for \( \Sigma \), any particular slice \( \Sigma \) could be embedded into the entire manifold in an infinite number of ways. The manner in which a slice is embedded can be described by the extrinsic curvature \( K_{AB} \) which describes how the spatial projection of the gradient of the surface normal \( n^A \) varies over the slice

\[
K_{AB} \equiv \gamma^C A \nabla_C n_B .
\]  

(A.1)

We define the projection \( \gamma_{AB} = g_{AB} - n_A n_B \). For a Gaussian normal coordinate orthogonal to the surface \( \Sigma \), the 5D metric is then locally given by (signature \((-+:+:+))

\[
ds^2_{(5)} = g_{\mu\nu} dx^\mu dx^\nu + dy^2 .
\]  

(A.2)

Since the congruence of curves determined by \( t^A \) is hypersurface orthogonal, the extrinsic curvature \( K_{AB} \) is symmetric, and therefore given by the Lie transport along the normal direction

\[
K_{AB} = \frac{1}{2} L_n g_{AB} .
\]  

(A.3)

In Hamiltonian dynamics, \( K_{AB} \) would be a kind of conjugate momentum. With the definition of the covariant derivative

\[
D_A \equiv \gamma^B_A \nabla_B \]  

(A.4)

we are able to decompose also the Riemann tensor:

- **Gauss equation relates to the curvature of the hypersurface:**

\[
R_{ABCD} = \nabla_A K_{BC} - \nabla_B K_{AC} .
\]  

(A.5)

- **Codazzi equation determines the change of \( K_{AB} \) along the hypersurface:**

\[
D_B K^B_A - D_A K = R_{BC} n^B_A n^C .
\]  

(A.6)

Other useful projections are

\[
(5) R_{EFGH} \gamma^E_A \gamma^F_B \gamma^G_C \gamma^H_D = \nabla_A K_{BC} - \nabla_B K_{AC} \]  

(A.7)

\[
(5) R_{EFGH} \gamma^E_A n^F_B n^G_C n^H_D = -L_n K_{AB} + K_{AC} K^A_B \]  

(A.8)

\[
(5) R_{CD} \gamma^C_A \gamma^D_B = R_{AB} - L_n K_{AB} - K K_{AB} + 2 K_{AC} K^C_B .
\]  

(A.9)
The above slicing also applies for the normal 3+1 slicing of a spacetime. Then one quite often needs the decomposition of Einstein’s equations. For this we also decompose the energy–momentum tensor into the ADM–density \( \rho \), a mass–current \( j^i \) and a stress–tensor \( S_{ab} \)

\[
\rho_{\text{ADM}} = n^a n^b T_{ab} \tag{A.10}
\]

\[
 j^b = n_a T^{ab} \tag{A.11}
\]

\[
 S_{ab} = \perp T_{ab} = \gamma_{ac} \gamma_{bd} T^{cd}. \tag{A.12}
\]

The decomposition of the Einstein tensor leads then to two constraint equations

\[
 G_{ab} n^a n^b = 8\pi G T_{ab} n^a n^b \tag{A.13}
\]

which is equivalent to

\[
 (3) R + K^2 − K_b^b K_a^a = 16\pi G \rho_{\text{ADM}}. \tag{A.14}
\]

The second constraint equation is the momentum equation

\[
 G^{ab} n_a = 8\pi G T^{ab} n_b, \tag{A.15}
\]

or

\[
 D_b K^{ab} − D^a K = 8\pi G j^a. \tag{A.16}
\]

Since the metric is written for the decomposition of the vector field \( t^a = \alpha n^a + \beta^a \) (Fig. 3.1),

\[
 ds^2 = -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt) (dx^j + \beta^j dt), \tag{A.17}
\]

we get the evolution equation for the 3–metric

\[
 L_t \gamma_{ab} = \alpha L_n \gamma_{ab} + L_\beta \gamma_{ab}
 = -2\alpha K_{ab} + L_\beta \gamma_{ab}. \tag{A.18}
\]

The time–evolution for the extrinsic curvature follows now from the \((n - 1)\)–dimensional Einstein equations

\[
 \perp G^{ab} = 8\pi G \perp T^{ab}, \tag{A.19}
\]

or

\[
 L_t K_b^a = L_\beta K_b^a − D^a D_b \alpha
 + \alpha \left[ (3) R_b^a + K K_b^a + 8\pi G \left( \frac{1}{2} \gamma_b^a (S^c_c - \rho_{\text{ADM}}) \right) - S_b^a \right]. \tag{A.20}
\]