Einstein’s Odyssee from Special Relativity to General Relativity

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"Things should be made as simple as possible, but not any simpler."
- Albert Einstein
From SR to GR

- Lorentz transformations
- Effects in SR
- Minkowski spacetime
- Causal structure of SR
- Maxwell’s theory in SR
- Hydrodynamics in SR
- How to incorporate gravity?
  ➔ Equivalence Principles
- SpaceTime as manifold of events
The “Æther” (ether) was then proposed
- A flexible substance enough to penetrate everything, yet rigid enough to be a medium for the high speed of light (Dark energy?)

How do we find the existence of the ether?
- In 1887, the Michelson-Morley experiment had a null-result

An explanation
- Lorentz proposed that space shrinks (or contracts) in the direction of travel through the ether by a factor of:

\[
\sqrt{1 - \left(\frac{v}{c}\right)^2}
\]
Galilean Relativity

• Choose a *frame of reference* (*i.e.* *a coordinate system*)
  - Necessary to describe physical events

• According to Galilean Relativity, the laws of mechanics are the same in all *inertial frames of reference* (*inertial frame* → *v*=constant)
  - An inertial frame of reference is one in which Newton’s Laws are valid
  - Objects subjected to no forces will move in straight lines
A Brief Review of Special Relativity

- **Special relativity (SR)** is the physical theory of measurement in inertial frames of reference proposed in 1905 by Albert Einstein (after considerable contributions of Hendrik Lorentz and Henri Poincaré) in the paper "On the Electrodynamics of Moving Bodies".

- It generalizes Galileo's principle of relativity – that all uniform motion is relative, and that there is no absolute and well-defined state of rest (no privileged reference frames) – from mechanics to all the laws of physics.

- In addition, special relativity incorporates the principle that the **speed of light** is the same for all inertial observers regardless of the state of motion of the source.
The Special Theory of Relativity

- Aimed to answer some burning questions:
  - Could Maxwell’s equations for electricity and magnetism be reconciled with the laws of mechanics?
  - Where was the aether?
2 Simple Postulates (1905)

- “The laws of physics are the same in every inertial frame of reference”
  - *The Principle of Relativity*

- “The speed of light in vacuum is the same in all inertial frames of reference, and is independent of the motion of the source”
  - *Invariance of the speed of light*
The Principle of Relativity

• This is a sweeping generalization of the principle of Galilean relativity, which refers only to the laws of Newtonian mechanics.
• The results of any kind of experiment performed in a laboratory at rest must be the same as when performed in a laboratory moving at a constant speed past the first one.
• No preferred inertial reference frame exists.
• It is impossible to detect absolute motion.
The principle of relativity, which states that there is no preferred inertial reference frame, dates back to Galileo, and was incorporated into Newtonian Physics. However, in the late 19th century, the existence of electromagnetic waves led physicists to suggest that the universe was filled with a substance known as "aether", which would act as the medium through which these waves, or vibrations traveled. The aether was thought to constitute an absolute reference frame against which speeds could be measured. In other words, the aether was the only fixed or motionless thing in the universe. Aether supposedly had some wonderful properties: it was sufficiently elastic that it could support electromagnetic waves, and those waves could interact with matter, yet it offered no resistance to bodies passing through it. The results of various experiments, including the Michelson-Morley experiment, indicated that the Earth was always 'stationary' relative to the aether – something that was difficult to explain, since the Earth is in orbit around the Sun.

Einstein's elegant solution was to discard the notion of an aether and an of absolute state of rest. Special relativity is formulated so as to not assume that any particular frame of reference is special; rather, in relativity, any reference frame moving with uniform motion will observe the same laws of physics. In particular, the speed of light in a vacuum is always measured to be \( c \), even when measured by multiple systems that are moving at different (but constant) velocities.
The classical aether theory underlying Maxwell’s equations has the aether at rest in some (unknown) inertial frame throughout space.

(We still teach and use Maxwell’s equations, without the underlying aether. How that happened is an interesting story beyond the scope of this presentation.)

Light propagates as a disturbance in the aether, isotropically with speed c relative to the aether frame.

As in Newtonian mechanics, Galilean relativity applies.

This theory has several fatal problems:

- **Maxwell’s equations** are not valid on earth, even though they were discovered here.
- By ~1900 about a dozen experiments failed to see effects related to the motion of the earth through the aether.
Earth moves relative to the Aether
Michelson-Morley Experiment
Lorentz Transformations

- Time and Space are correlated
- Time and Space are symmetric
- Inverse transformation

\[
\begin{align*}
t' &= \gamma (t - vx) \\
x' &= \gamma (x - vt) \\
y' &= y \\
z' &= z
\end{align*}
\]
Lorentz Transformation (derivation)

Look for a transformation of the form
\[ ct' = Act + Bx \]
\[ x' = Dx + Ect \]
\[ y' = y \]
\[ z' = z \]

where A, B, D, and E are to be determined. The inverse transformation is given by
\[ ct = (Dct' - Bx') / (AD - BE) \]
\[ x = (Ax' - Ect') / (AD - BE) \]
\[ y = y' \]
\[ z = z' \]
Also we know that for $x' = 0 \ dx/dt = V$ and for $x = 0 \ dx'/dt' = -V$ thus

$$V = \frac{dx}{dt} = -\frac{E}{D}c$$
and

$$V = -\frac{dx'}{dt'} = -\frac{E}{A}c$$

and hence $D = A$ and $E = -AV/c = -A\beta$ with $\beta = V/c$. We now require that

$$(ct)^2 - x^2 = (ct')^2 - x'^2 = (Act + Bx)^2 - (Ax + Ect)^2$$

which implies that

$$(1 - A^2 + E^2)(ct)^2 - (1 - A^2 + B^2)x^2 + A(E - B)2xct = 0$$

and hence $B = E$ and $1 - A^2 + E^2 = 1 - A^2 + A^2\beta^2 = 0$ so that $A^2 = 1/(1 - \beta^2)$.

Thus,

$$A = \gamma = 1/\sqrt{1 - \beta^2}$$

$$B = E = -\beta\gamma$$

$$D = A = \gamma$$

$$E = -\beta\gamma$$
Lorentz Transformations

- Vector notation for events \((\mu, \nu=0, \ldots, 3)\)

\[
\begin{pmatrix}
  x'^0 \\
  x'^1 \\
  x'^2 \\
  x'^3
\end{pmatrix}
= \begin{pmatrix}
  \gamma & -\nu\gamma & 0 & 0 \\
  -\nu\gamma & \gamma & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x^0 \\
  x^1 \\
  x^2 \\
  x^3
\end{pmatrix}
\]

\[x'{}^{\mu} = \Lambda^{\mu}{}_{\nu} x^\nu\]
Boost in arbitrary Direction

\[
\begin{bmatrix}
ct' \\
x' \\
y' \\
z'
\end{bmatrix} = \\
\begin{bmatrix}
\gamma & -\beta_x \gamma & -\beta_y \gamma & -\beta_z \gamma \\
-\beta_x \gamma & 1 + (\gamma - 1)\frac{\beta_x^2}{\beta^2} & (\gamma - 1)\frac{\beta_x \beta_y}{\beta^2} & (\gamma - 1)\frac{\beta_x \beta_z}{\beta^2} \\
-\beta_y \gamma & (\gamma - 1)\frac{\beta_y \beta_x}{\beta^2} & 1 + (\gamma - 1)\frac{\beta_y^2}{\beta^2} & (\gamma - 1)\frac{\beta_y \beta_z}{\beta^2} \\
-\beta_z \gamma & (\gamma - 1)\frac{\beta_z \beta_x}{\beta^2} & (\gamma - 1)\frac{\beta_z \beta_y}{\beta^2} & 1 + (\gamma - 1)\frac{\beta_z^2}{\beta^2}
\end{bmatrix} \begin{bmatrix}
ct \\
x \\
y \\
z
\end{bmatrix},
\]
Consequences of Special Relativity

- Restricting the discussion to concepts of length, time, and simultaneity.
- In relativistic mechanics:
  - There is no such thing as absolute length.
  - There is no such thing as absolute time.
  - Events at different locations that are observed to occur simultaneously in one frame are not observed to be simultaneous in another frame moving uniformly past the first.
• **Time dilation** – the time lapse between two events is not invariant from one observer to another, but is dependent on the relative speeds of the observers' reference frames.

• **Relativity of simultaneity** – two events happening in two different locations that occur simultaneously to one observer, may occur at different times to another observer (lack of absolute simultaneity).

• **Lorentz contraction** – the dimensions (e.g., length) of an object as measured by one observer may be smaller than the results of measurements of the same object made by another observer.

• **Composition of velocities** – velocities (and speeds) do not simply 'add', for example if a rocket is moving at \( \frac{2}{3} \) the speed of light relative to an observer, and the rocket fires a missile at \( \frac{2}{3} \) of the speed of light relative to the rocket, the missile does not exceed the speed of light relative to the observer.

• **Inertia and momentum** – as an object's speed approaches the speed of light from an observer's point of view, its mass appears to increase thereby making it more and more difficult to accelerate it from within the observer's frame of reference.

• **Equivalence of mass and energy, \( E = mc^2 \)** – Conservation of energy implies that in any reaction a decrease of the sum of the masses of particles must be accompanied by an increase in kinetic energies of the particles after the reaction.
Velocity Addition Theorem

\begin{align*}
    u'_x &= \frac{u_x - v}{1 - vu_x/c^2} \\
    u'_y &= \frac{u_y}{\gamma(1 - vu_x/c^2)} \\
    u'_z &= \frac{u_z}{\gamma(1 - vu_x/c^2)}.
\end{align*}

Velocities are not simply superpositions
• Useful Four-Vectors

\[ x^\mu = \begin{pmatrix} t \\ \mathbf{r} \end{pmatrix} \]

\[ U^\mu = \frac{dx^\mu}{d\tau} = \begin{pmatrix} \gamma_u \\ \gamma_u \mathbf{u} \end{pmatrix} \quad p^\mu = m \begin{pmatrix} \gamma_u \\ \gamma_u \mathbf{u} \end{pmatrix} \]

\[ k^\mu = \begin{pmatrix} \omega \\ \mathbf{k} \end{pmatrix} \quad p^\mu = \begin{pmatrix} \hbar \omega \\ \hbar \mathbf{k} \end{pmatrix} \]
Minkowski Line Element

Event: (ct, x) →
Spacetime = set of all events

$$ds^2 = c^2 dt^2 - dl^2,$$

Spherical Polar Coordinates

$$dl^2 = dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$dl$ is distance between two nearby points

- $x = r \sin \theta \cos \phi$
- $y = r \sin \theta \sin \phi$
- $z = r \cos \theta$
At each event, a lightcone is defined.
The light cone

Think of the light cone as the surface of an expanding sphere of light.
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The light cone

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Think of the light cone as the surface of an expanding sphere of light.
The forward light cone includes all places that can receive light from Star's origin.
The past light cone includes all places that can send light to Observer’s origin.
Invariance of Minkowski Metric

\[ ds^2 = (dx)^T \eta (dx) = (dx')^T \eta (dx') \]

\[ \eta = \Lambda^T \eta \Lambda, \quad (dx') = \Lambda (dx), \quad (dx) = (dx^0, dx^i)^T \]

The matrices which satisfy (2.13) are known as the Lorentz transformations; the set of them forms a group under matrix multiplication, known as the Lorentz group. There is a close analogy between this group and \( O(3) \), the rotation group in three-dimensional space. The rotation group can be thought of as \( 3 \times 3 \) matrices \( R \) which satisfy

\[ I = R^T \cdot I \cdot R, \quad (2.14) \]
The Lorentz Group

\[ \det(\Lambda^T) \cdot \det(\eta) \cdot \det(\Lambda) = \det(\eta). \]

Since \( \det(\eta) = -1 \) and \( \det(\Lambda^T) = \det(\Lambda) \), it follows

\[ \det^2(\Lambda) = +1, \quad \det(\Lambda) = \pm 1. \]

If the 00–element of (2.13) is written out, it gives

\[ (\Lambda^T \cdot \eta \cdot \Lambda)_0^0 = \eta_{00} = 1. \]

This can be written as

\[ (\Lambda_0^0)^2 = 1 + (\Lambda_1^0)^2 + (\Lambda_2^0)^2 + (\Lambda_3^0)^2. \]
The Lorentz group is a subgroup of the Poincaré group, the group of all isometries of Minkowski spacetime. The Lorentz transformations are precisely the isometries which leave the origin fixed. Thus, the Lorentz group is an isotropy subgroup of the isometry group of Minkowski spacetime.
LT as Pseudo-Rotations

\[ ct' = ct \cosh \Phi - x \sinh \Phi \]
\[ x' = -ct \sinh \Phi + x \cosh \Phi. \]

From this we see that the point defined by \( x' = 0 \) is moving with a velocity

\[ \beta = \frac{v}{c} = \frac{x}{ct} = \frac{\sinh \Phi}{\cosh \Phi} = \tanh \Phi. \]

To translate into more pedestrian notation, we can replace \( \Phi = \tanh^{-1}(v/c) \) and then

\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} = \cosh \Phi \]
\[ \gamma \beta = \sinh \Phi \]

to obtain the well-known classical expressions for the Lorentz transformations.
Velocity Addition Theorem

produces also a Lorentz transformation. Using pseudo-rotations we find

\[
ct' = ct \cosh \Phi_1 - x \sinh \Phi_1
\]  \hspace{1cm} \text{(3.35)}

\[
x' = -ct \sinh \Phi_1 + x \cosh \Phi_1
\]  \hspace{1cm} \text{(3.36)}

\[
t' = ct' \cosh \Phi_2 - x' \sinh \Phi_2 = ct \cosh \Phi - x \sinh \Phi
\]  \hspace{1cm} \text{(3.37)}

\[
x'' = -ct' \sinh \Phi_2 + x' \cosh \Phi_2 = -ct \sinh \Phi + x \cosh \Phi,
\]  \hspace{1cm} \text{(3.38)}

with \( \Phi = \Phi_1 + \Phi_2 \). With \( \tanh \Phi = v/c \) and the well-known theorem for the hyperbolic tangent

\[
\tanh(\Phi_1 + \Phi_2) = \frac{\tanh \Phi_1 + \tanh \Phi_2}{1 + \tanh \Phi_1 \tanh \Phi_2}
\]  \hspace{1cm} \text{(3.39)}

we obtain for the combined velocity

\[
v = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}.
\]  \hspace{1cm} \text{(3.40)}
Lorentz Tensors

- Tensors transform covariantly under LTs

\[ T'_{\mu \nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha \beta} \]

\[ T^{\alpha \beta}_{\nu} \Leftrightarrow A^\alpha B^\beta C_\nu \]

\[ A_{,\mu} = \frac{\partial A}{\partial x^\mu} \quad A^{,\mu} = \frac{\partial A}{\partial x_\mu} \]
Higher Rank Tensors

\[ T^{\mu \nu} \rightarrow T'{}^{\mu \nu} = \Lambda^\mu_\rho \Lambda^\nu_\sigma T^{\rho \sigma}. \]

\[ T = (\rho c^2 + P) \underline{u} \otimes \underline{u} + P \underline{g}, \]

\[ \varepsilon^{\alpha \beta \gamma \delta} = \begin{cases} +1 & , \alpha \beta \gamma \delta \text{ even permutation of } 0123 \\ -1 & , \alpha \beta \gamma \delta \text{ odd permutation of } 0123 \\ 0 & , \text{otherwise} \end{cases} \]

\[ \Lambda^\alpha_\epsilon \Lambda^\beta_\zeta \Lambda^\gamma_\kappa \Lambda^\delta_\lambda \varepsilon^{\epsilon \zeta \kappa \lambda} = +\varepsilon^{\alpha \beta \gamma \delta} \]

\( \Rightarrow \text{Dual Tensors} \)
Motivation for Tests of SR

- It is worthwhile to occasionally check the basics.
- Special Relativity (SR) is part of the foundation of every mainstream theory of physics today.
- The quest for quantum gravity has inspired a search for ways SR might be modified in a consistent manner.
SR makes many predictions, which are well tested:

- Isotropy of the speed of light
- Isotropy of space
- Constancy of the speed of light
- Time dilation and Doppler
- Length contraction
- Twin paradox
- Relativistic kinematics
- Relativistic velocity addition
- Variation of c with frequency
- g-2 as test of SR
- Other – 14
The isotropy of c is particularly well tested:
- Michelson-Morley (and variations) – 14
- Laser/Maser tests – 8
- Atomic beams – 2
- Frequency-doubling interferometer
- Cryogenic optical resonators – 4
- One-Way tests
  - Two lasers – 6
  - Two atomic clocks – 3
  - Rotating Mössbauer absorbers – 4
Michelson – Morley Experiment (1887)

- Finicky experiment: ±0.002 °C, mechanical stability ~nm
- Result: upper limit of 7.5 km/s (earth relative to aether)
• Vastly less finicky than Michelson-Morley
  – Invar components with low thermal expansion
  – Rotating Fabry-Perot etalon is vacuum
  – Uses frequency (motion 1 wavelength/sec => 1 Hz, ~1 part in $10^{15}$)
• Result: $\Delta f/f = (1.5\pm2.5) \cdot 10^{-15}$ $\Rightarrow$ 0.02 km/s ($V_{\text{earth}}$)
As a simple test theory, assume the observed speed of light is given by

\[ V_{\text{obs}} = c + k V_{\text{source}} \]

with \( k \) to be determined by experiment.

A test at CERN using \( \pi^0 \) decay:

\[ k < 4 \cdot 10^{-4} \]

Distant supernovas have a velocity spread of the remnants \( \sim 10,000 \) km/s (obtained via Doppler broadening). Observations of supernovas \( \sim 5 \) billion lightyears away show the light reaches us within \( \sim 10 \) days:

\[ k < 10^{-9} \]
The Solar System: Hierarchy of Celestial Frames

- **Solar System Barycentric Frame**
- **Heliocentric Frame**
- **Geocentric Frame**
- **Lunocentric Frame**

Planets and Moons:
- Saturn
- Jupiter
- Mars
- Moon
- Mercury
- Venus
- Earth
- Neptune
- Pluto & Charon
Vectors and Tensors

Vectors and tensors

Four-vectors

\[ A^\mu \equiv (A^0, \vec{A}) \quad (\mu = 0, 1, 2, 3) \]

Scalar product (Lorentz-invariant!)

\[ A \cdot B = A^\mu B_\mu = A^0 B^0 - \vec{A} \cdot \vec{B} = \eta^{\mu \nu} A_\mu B_\nu \]
Minkowski metric

(cartesian coordinates)

\[
\eta_{\mu \nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

Gradient operator:

\[
\nabla_{\mu} = \left( \frac{1}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)
\]

\[
\nabla_{\mu} = \eta^{\mu \nu} \nabla_{\nu} = \left( \frac{1}{c} \frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right)
\]
4-Velocity and 4-Momentum

Four-position
('worldline')

\[ x^\mu = (ct, \vec{x}) \]

Four-momentum

\[ p^\mu = \left( \frac{E}{c}, \vec{p} \right) = mU^\mu \]

4-velocity & 4-acceleration

\[ u^\mu \rightarrow u'^\mu = \Lambda^\mu_\nu u^\nu. \]

\[ a^\mu = \frac{dw^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}. \]

\[ U^\mu = \Gamma(c, \vec{v}) = \frac{(c, \vec{v})}{\sqrt{1 - v^2/c^2}} \]

\[ U \cdot U = U^\mu U_\mu = c^2 \]
Worldline's 4-acceleration

The 4-acceleration of the particle

\[ \vec{a} = \nabla_{\vec{u}} \vec{u} \, . \]

Since \( \vec{u} \) is a unit vector, it follows that

\[ \vec{u} \cdot \vec{a} = 0 \, , \]

4-velocity \( \vec{u} \) is timelike, 4-acceleration \( \vec{a} \) is a spacelike vector.
Relativistic Momentum

- To account for conservation of momentum in all inertial frames, the definition must be modified

\[ p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma mv \]

- \( v \) is the speed of the particle, \( m \) is its mass as measured by an observer at rest with respect to the mass

- When \( v \ll c \), the denominator approaches 1 and so \( p \) approaches \( mv \)
Relativistic Energy

• The definition of kinetic energy requires modification in relativistic mechanics

• $KE = \gamma mc^2 - mc^2$
  - The term $mc^2$ is called the rest energy of the object and is independent of its speed
  - The term $\gamma mc^2$ is the total energy, $E$, of the object and depends on its speed and its rest energy
Energy and Relativistic Momentum

• It is useful to have an expression relating total energy, $E$, to the relativistic momentum, $p$
  - $E^2 = p^2c^2 + (mc^2)^2$
    - When the particle is at rest, $p = 0$ and $E = mc^2$
    - Massless particles ($m = 0$) have $E = pc$
  - This is also used to express masses in energy units
    - mass of an electron = $9.11 \times 10^{-31}$ kg = 0.511 MeV
    - Conversion: 1 u = 929.494 MeV/c$^2$
The Laws of Physics

- Write them as **tensor equations** (tensors are Lorentz covariant entities).
- **E** and **B** fields in Maxwell’s theory e.g. are not covariant ➞ use **Faraday tensor**.

стеребрено ➞ **Use conservation of energy and momentum.**

стеребрено ➞ **Derive field equations, if possible, from Lagrangians.**
Euler-Lagrange equations

$$\frac{\delta S}{\delta \phi} = -\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) + \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$
Hydrodynamic Equations
Newtonian Euler Equations
Density, Velocity, Energy

\[ \partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \]
\[ \rho \left( \partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla P - \rho \nabla \Phi \]
\[ \partial_t \left( \frac{\rho \vec{v}^2}{2} + \rho e \right) = -\nabla \cdot \left[ \rho \vec{v} \left( \frac{\vec{v}^2}{2} + h \right) \right] \]
\[ \nabla^2 \Phi = 4\pi G \rho , \]
3 Conservation Laws

Basic Equations without gravity

Mass conservation:
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \]

Momentum conservation:
\[ \frac{\partial}{\partial t} (\rho \vec{V}) + \nabla \cdot (\rho \vec{VV} + P\vec{I}) = 0 \]

Energy conservation:
\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \rho V^2 + \rho \epsilon \right) + \nabla \cdot \left[ \rho \vec{V} \left( \frac{1}{2} V^2 + h \right) \right] = 0 \]

Specific enthalpy
Hydro State Vector and Fluxes

State vector: $U = (\rho, \rho \, v_k, \rho \, [v^2/2 + e])^T$ : 5D

Flux vector:
$$F^i = (\rho v^i, \rho v_k v^i + P \delta^i_k, [v^2/2 + h] \rho v^i)^T$$

Conservative formulation of hydrodynamics
Relativistic Mass Conservation

**Scalar density:**  
mass conservation

\[ \nabla_{\mu} J^\mu = 0 \, , \, J^\mu = n m U^\mu \]

**Lorentz-Invariant:**

\[ J^\mu U_\mu = n m c^2 \]

\[ \frac{\partial}{\partial t} (\Gamma n m) + \vec{\nabla} \cdot (\Gamma n m \vec{V}) = 0 \]

**3+1 split:**

\[ \frac{\partial \rho}{\partial t} + (\vec{\nabla} \cdot \rho \vec{V}) = 0 \quad \text{with} \quad \rho \equiv \Gamma n m \]

**Lorentz-contraction**
Energy-Momentum Conservation

Generic Form:

\[ \nabla_\mu T^{\mu \nu} = 0 \iff 1 \frac{\partial T^{0 \nu}}{c \partial t} + \frac{\partial T^{i \nu}}{\partial x^i} = 0 \]

\( i = 1, 2, 3 \)

\( \nu = 0 \) should reduce to energy conservation;

\( \nu = 1, 2, 3 \) should reduce to momentum conservation.
Decompose Energy-Momentum Tens

\[ T = (\rho c^2 + P) \mathbf{u} \otimes \mathbf{u} + Pg, \]

Total energy

\[ E = W^2(\rho c^2 + P) - P. \]

Momentum flux

\[ \mathbf{\tilde{S}} = W^2 \left( \rho + \frac{P}{c^2} \right) \mathbf{\tilde{U}}. \]

Stress tensor

\[ S_{ij} = P \delta_{ij} + W^2 \left( \rho + \frac{P}{c^2} \right) V^i V^j, \]

Observer velocity

\[ \mathbf{\tilde{u}} \cdot \mathbf{\tilde{u}}_0 = -W \]

\[ E = T(\mathbf{\tilde{u}}_0, \mathbf{\tilde{u}}_0) \]

\[ c\mathbf{\tilde{S}} = - T(\mathbf{\tilde{u}}_0, \mathbf{\tilde{e}}_i) \mathbf{\tilde{e}}_i \]

\[ S_{ij} = T(\mathbf{\tilde{e}}_i, \mathbf{\tilde{e}}_j) \]
Hydro Conservation Laws

State vector \( U = (D, S_i, \tau)^T \)

Primitive variables \( P = (\rho, v_1, v_2, v_3, P)^T \)

\[
D = W \rho_0 \\
\vec{S} = \rho_0 W^2 h \vec{v} \\
\tau = \rho_0 W^2 h - P - D = E - D
\]

Fluxes:
\[
F^i = \left( Dv^i, S_j v^i + P \delta^i_j, (\tau + P)v^i \right)
\]
An integral conservation law asserts that the rate of change of the total amount of a quantity with density \( u \) in a fixed control volume \( V \) is equal to the total flux of the quantity through the boundary \( dV \)

\[
\frac{\partial}{\partial t} \int_V u \, dV + \int_{\partial V} f(u) \cdot dA = 0
\]

The integral conservation law is transferred to small control volumes.
Method of Finite Volumes

\[ \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0. \]

We take the volume integral over the total volume of a cell, called control volume, \( v_i \),

\[ \int_{v_i} \frac{\partial \mathbf{u}}{\partial t} \, dv + \int_{v_i} \nabla \cdot \mathbf{f}(\mathbf{u}) \, dv = 0. \]

\[ v_i \frac{d\mathbf{u}_i}{dt} + \oint_{S_i} \mathbf{f}(\mathbf{u}) \, dS = 0, \]

\[ \frac{d\mathbf{u}_i}{dt} + \frac{1}{v_i} \oint_{S_i} \mathbf{f}(\mathbf{u}) \, dS = 0. \]

\( i = 1, \ldots, N_{\text{Cells}} \)

\( N_{\text{Cells}} \sim 1024^3 \)
3D Rayleigh-Taylor Instability

Gravity

FLASH-code

Density (g/cc) t = 3.1 sec

3D Rayleigh-Taylor Instability
Space-time picture Quark-Gluon Plasma (RHIC; LHC)

Bjorken formula

\[ s(\tau_0 = 1 \text{ fm/c}) \approx 30/\text{fm}^3 \]
or
\[ T(\tau_0) \approx 275 \text{ MeV} \]
in Au+Au (200 GeV)
Modern Finite Volume Techniques

- Stable and sharp discrete shock profiles
- Accurate propagation speed of discontinuities
- Accurate resolution of multiple nonlinear structures: discontinuities, rarefaction waves, vortices, etc

Simulation of an extragalactic relativistic jet

Wind accretion onto a Kerr black hole (a= 0.999M)
MAXWELL’S EQUATIONS for the VACUUM (cgs, often c=1)

\[ \nabla \cdot \vec{E} = 4\pi \rho, \quad (3), \quad \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \quad (12) \]

\[ \nabla \cdot \vec{B} = 0, \quad (14) \quad \nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}, \quad (13) \]

Continuity equation

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0, \quad (1) \]
Electromagnetic Field

- Electromagnetic Faraday Tensor

\[
F_{\mu \nu} = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & B_z & -B_y \\
E_y & -B_z & 0 & B_x \\
E_z & B_y & -B_x & 0
\end{pmatrix}
\]
Electromagnetic Field

• Covariance of Electrodynamics
  – Invariance of Maxwell’s Equations by transformation between inertial frames
  – Tensors are Lorentz covariant

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \Rightarrow j_{\mu}^{\ ,\mu} = 0
\]

\[
\nabla \cdot \mathbf{E} = 4\pi \rho, \quad \nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{j} \Rightarrow \quad F_{\mu\nu}^{\ ,\nu} = \frac{4\pi}{c} j_{\mu}
\]
Faraday and its Dual Tensor

\[ F_{\mu\nu} = \begin{pmatrix}
0 & -E_x/c & -E_y/c & -E_z/c \\
E_x/c & 0 & -B_z & B_y \\
E_y/c & B_z & 0 & -B_x \\
E_z/c & -B_y & B_x & 0
\end{pmatrix} \]

\[ \tilde{F}^{\mu\nu} := \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \implies \tilde{F}^{\alpha\beta} = \begin{bmatrix}
0 & -B_x & -B_y & -B_z \\
B_x & 0 & E_z/c & -E_y/c \\
B_y & -E_z/c & 0 & E_x/c \\
B_z & E_y/c & -E_x/c & 0
\end{bmatrix} \]

\[ \partial_\mu F^{\mu\nu} = \frac{1}{c} j^\nu \quad \partial_\mu \tilde{F}^{\mu\nu} = 0 \]
Energy-Momentum Tensor

\[ T^\mu\nu = -\frac{1}{\mu_0} \left[ F^\mu\alpha F_{\alpha\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right]. \]

\[
T^\mu\nu = \begin{bmatrix}
\frac{1}{2} (\epsilon_0 E^2 + \frac{1}{\mu_0} B^2) & \frac{S_x}{c} & \frac{S_y}{c} & \frac{S_z}{c} \\
\frac{S_x}{c} & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\
\frac{S_y}{c} & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\
\frac{S_z}{c} & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz}
\end{bmatrix}
\]

\[
\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}
\]

\[
\sigma_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2} \left( \epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \delta_{ij}
\]
EM Conservation Laws

\[ \partial_\nu T^{\mu\nu} + \eta^{\mu\rho} f_\rho = 0 \]

\[ \frac{\partial u_{em}}{\partial t} + \nabla \cdot \vec{S} + \vec{J} \cdot \vec{E} = 0 \]

\[ \frac{\partial \vec{p}_{em}}{\partial t} - \nabla \cdot \sigma + \rho \vec{E} + \vec{J} \times \vec{B} = 0 \]

\[ u_{em} = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \quad \vec{p}_{em} = \frac{\vec{S}}{c^2} \]
Electromagnetic Field

- Electromagnetic Field Transformations

\[
F'_{\mu \nu} = \tilde{\Lambda}^{\alpha}_{\mu} \tilde{\Lambda}^{\beta}_{\nu} F_{\alpha \beta}
\]

\[
E'_{\parallel} = E_{\parallel} \quad E'_{\perp} = \gamma (E_{\perp} + v \times B)
\]

\[
B'_{\parallel} = B_{\parallel} \quad B'_{\perp} = \gamma (B_{\perp} - v \times E)
\]
• **Special Relativity** is well established.
• Invariance of the speed of light is well tested, no preferred frame of reference in Minkowski.
• Laws of physics are to be written in covariant way: ➔ Maxwell’s theory with Faraday tens, ➔ Lagrangian field theories (scalar), ➔ Motion of a perfect fluid is written as a set of 3 conservation laws (numerically import).
• Open question: **How to include gravity?**