The Inflationary Universe

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Modern Cosmology
Nov 2010 @ Day 6/2
Big Bang
t = 0

Quark-Gluon Plasma

End of Planck time; gravity freezes out
Strong force freezes out; inflation begins
Weak and electromagnetic forces freeze out
Confinement (of quarks)
Universe transparent to neutrinos
Synthesis of primordial helium
Universe transparent to photons
Now

Temperature (K)

Quantized gravity

Inflationary epoch

Time after Big Bang (s)
Why do we Need Inflation?

Problems of the standard Big Bang theory:

• What was before the Big Bang?
• Why is our Universe so **homogeneous** (better than 1 part in 10000) ?
• Why is it **isotropic** (the same in all directions)?
• Why all of its parts started expanding simultaneously?
• Why it is **flat**? Why parallel lines do not intersect? Why it contains so many particles? Why there are so many people in this auditorium?
Two New Cosmological Discoveries

• (i) The new-born universe experienced rapid acceleration (called Inflation)

• (ii) A new (slow) stage of acceleration started 5 billion years ago (Dark Energy)

Two Major questions:

How did the Universe start, and how is it going to end?
Topics

- What is Inflation?
- On Inflation History: 1981 Alan Guth, ...
- **5 Problems** of the Standard Model
- How to describe Inflation? – The **Inflaton Field** and Slow-Roll Conditions.
- Fluctuations in the Inflaton field.
- Evolution of the Perturbations.

▸ Universal Fluctuation Spectrum
In physical cosmology, **cosmic inflation**, **cosmological inflation** or just **inflation** is an exponential expansion of the Universe at the end of the grand unification epoch, ~10^{-38} seconds after the Big Bang, driven by a **negative-pressure vacuum energy density**. The term "inflation" is also used to refer to the hypothesis that inflation occurred, to the theory of inflation, or to the **inflationary epoch**.
Idea of Inflationary Universe
As a direct consequence of this expansion, all of the observable universe originated in a small causally connected region. Inflation answers the classic conundrum of the big bang cosmology: why does the universe appear flat, homogeneous and isotropic in accordance with the cosmological principle when one would expect, on the basis of the physics of the big bang, a highly curved, heterogeneous universe? Inflation also explains the origin of the large-scale structure of the cosmos. Quantum fluctuations in the microscopic inflationary region, magnified to cosmic size, become the seeds for the growth of structure in the universe (see galaxy formation and evolution and structure formation).
Inflation was proposed by Alexei Starobinski (1979/80) in the Soviet Union, and simultaneously by Alan Guth (1980/81) in the United States. Guth's mechanism is different from Starobinski's, and requires a modification to allow for a graceful exit from inflation. This modification was provided independently by Andrei Linde, and by Andreas Albrecht and Paul Steinhardt.
### Cosmological Phase Transitions

<table>
<thead>
<tr>
<th>Phase Transition</th>
<th>Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>QCD phase transition</td>
<td>$T = 175$ MeV</td>
</tr>
<tr>
<td>Electroweak phase transition</td>
<td>$T = 150$ GeV</td>
</tr>
<tr>
<td>Baryogenesis</td>
<td>$T = 10^{16}$ GeV</td>
</tr>
<tr>
<td>GUT phase transition(s) ?</td>
<td>$T = 10^{16}$ GeV</td>
</tr>
<tr>
<td>Monopoles, cosmic strings ?</td>
<td>$T = 10^{15}$ GeV</td>
</tr>
<tr>
<td>&quot;Inflation&quot;</td>
<td>$T = 10^{15}$ GeV</td>
</tr>
<tr>
<td>Primordial density fluctuations !</td>
<td></td>
</tr>
<tr>
<td>Primordial magnetic fields ?</td>
<td></td>
</tr>
</tbody>
</table>
Order of the Phase Transition

Temperature dependence of order parameter (magnetisation)
Guth’s Scenario 1981: GUT Phase Transition

As the universe cools, the true minimum gets deeper. At $T \sim T_s$, the universe tunnels to the “True Vacuum”.

Universe lingers in the so-called metastable “False Vacuum” with energy density $\rho_0$.

**Fig. 1.** Effective potential vs $\phi$ for various values of $T$.

$$\rho = \frac{\pi^2}{30} g(T) T^4 + \rho_0$$

Fig. A. Albrecht and P. Steinhardt, Phys. Rev. D, **48**, 1220 (1981).
What went wrong with Guth’s Theory?

Guth’s phase transition scenario surely conforms to these requirements. But there was a serious problem.

- The bubbles of true vacuum will form at various times in various places in the false vacuum.
- They will have great difficulty merging because the space separating them will still be expanding exponentially.
- The phase transition will never be completed even if the bubbles grow at the speed of light.

Guth & Weinberg (1983)
What came after Guth’s Theory?

• New Inflation (Linde 1982, Albrecht & Steinhardt 1982):
  Roll down a flat potential.

• Chaotic Inflation (Linde 1983)
  Slide through a viscous medium.
Chaotic Inflation 1983

Simple!!

No supercooling/tunneling from false vacuum. No plateau.

No thermal equilibrium!

In $10^{-35}$ s, a Plank size region blows up to $\sim 10^{10^{12}}$ cms!

⇒ Nonsense!

Initial Universe may be thought of as having chaotically distributed values of field. Inflation took place only where $\phi$ was large.

At the end of inflation, the field oscillates and decays into particle-pairs, which interact and thermalize at some temperature. Standard Big Bang takes over from here.
Proposed by Guth in 1981 to solve:
- Horizon problem
- Flatness problem

Basic idea: Universe undergoes exponential expansion in early history
Theories of Inflation 2010

1980
- $R^2$-inflation
- Old Inflation
  - New Inflation
  - Chaotic inflation
  - Double Inflation
  - Power-law inflation
  - SUGRA inflation
  - Extended inflation

1990
- Natural inflation
  - Hybrid inflation
  - SUSY F-term inflation
  - SUSY D-term inflation
  - Assisted inflation
  - Brane inflation

2000
- SUSY P-term inflation
- Super-natural Inflation
- N-flation
- K-flation
- DBI inflation
- Warped Brane inflation
- Tachyon inflation
- $D3-D7$ inflation
- Racetrack inflation

Roulette inflation Kahler moduli/axion
The Standard Big-Bang Model has many deep severe problems:

- **Scale Problem**: $\sim \mu m$ is no natural scale for $R_H$.
- **Flatness Problem**: The flatness problem is an observational problem associated with a FRW model: Why is $\Omega_k \sim 0$?
- **Causality Problem**: The causality or horizon problem results from the premise that information cannot travel faster than light.
- **Monopole Problem**: Grand unification theories predicted topological defects in space that would manifest as magnetic monopoles.
- **Structure Problem**: Structure formation not explained $\rightarrow$ no natural mechanism!
## The Scale Problem

<table>
<thead>
<tr>
<th>Event</th>
<th>Temp</th>
<th>Redshift</th>
<th>Time</th>
<th>Today’s size</th>
</tr>
</thead>
<tbody>
<tr>
<td>Now (1 Hubble Radius)</td>
<td>2.725 K</td>
<td>0</td>
<td>13.7 Gyr</td>
<td>4200 Mpc</td>
</tr>
<tr>
<td>Distant Galaxy</td>
<td>19 K</td>
<td>6</td>
<td>1 Gyr</td>
<td>600 Mpc</td>
</tr>
<tr>
<td>Recombination</td>
<td>3000 K</td>
<td>10^80</td>
<td>380,000 yr</td>
<td>4 Mpc</td>
</tr>
<tr>
<td>Radiation Dominance</td>
<td>9500 K</td>
<td>10^10</td>
<td>80,000 yr</td>
<td>1.3 Mpc</td>
</tr>
<tr>
<td>Nucleosynthesis</td>
<td>10^{10} K</td>
<td>10^{10}</td>
<td>1 sec</td>
<td>1.3 lyrs</td>
</tr>
<tr>
<td>Quark–Hadron Transition</td>
<td>150 MeV</td>
<td>10^{12}</td>
<td>10 μsec</td>
<td>1 lday</td>
</tr>
<tr>
<td>Electroweak Unification</td>
<td>300 GeV</td>
<td>10^{15}</td>
<td>10^{-12} sec</td>
<td>0.1 hours</td>
</tr>
<tr>
<td>SUSY Scale (?)</td>
<td>1 TeV</td>
<td>10^{16}</td>
<td>10^{-14} sec</td>
<td>lsecs</td>
</tr>
<tr>
<td>Grand Unification</td>
<td>10^{14} GeV</td>
<td>10^{28}</td>
<td>10^{-35} sec</td>
<td>1 cm</td>
</tr>
<tr>
<td>Inflation</td>
<td>10^{16} GeV</td>
<td>10^{30}</td>
<td>10^{-38} sec</td>
<td>1 mm</td>
</tr>
<tr>
<td>Quantum Gravity</td>
<td>10^{19} GeV</td>
<td>10^{32}</td>
<td>10^{-43} sec</td>
<td>1 μm</td>
</tr>
</tbody>
</table>
Flatness Problem: Why is Universe so flat?

- a multi-component universe satisfies (see Friedmann Equ)

\[
1 - \Omega(t) = - \frac{k c^2}{H(t)^2 a(t)^2 R_0^2} = \frac{H_0^2(1 - \Omega_0)}{H(t)^2 a(t)^2}
\]

and, neglecting \( \Lambda \),

\[
\left( \frac{H(t)}{H_0} \right)^2 = \frac{\Omega_{r0}}{a^4} + \frac{\Omega_{m0}}{a^3}
\]

therefore

- during radiation dominated era \(|1 - \Omega(t)| \propto a^2\)
- during matter dominated era \(|1 - \Omega(t)| \propto a\)
- if \(|1 - \Omega_0| < 0.02\) (WMAP),

then at CMB emission \(|1 - \Omega| < 0.00002\)

- we have a fine tuning problem!
When we look at the CMB it comes from 46 billion comoving light years away. However when the light was emitted the Universe was much younger (380,000 years old). In that time light would have only reached as far as the smaller circles. The two points indicated on the diagram would not have been able to contact each other because their spheres of causality do not overlap.
The Horizon Problem in Conformal Coordinates

\[ ds^2 = a^2(\eta) \left[ -c^2 d\eta^2 + dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right] \]
The Monopole Problem

Big issue in early 1980s

- Grand Unified Theories of particle physics → at high energies the strong, electromagnetic and weak forces are unified.
- the symmetry between strong and electroweak forces ‘breaks’ at an energy of \( \sim 10^{15} \text{ GeV} \) (\( T \sim 10^{28} \text{ K} \), \( t \sim 10^{-36} \text{ s} \))
  - this is a phase transition similar to freezing of water.
  - expect to form ‘topological defects’ (like defects in crystals).
  - point defects act as magnetic monopoles and have mass \( \sim 10^{15} \text{ GeV/c}^2 \) \( (10^{-12} \text{ kg}) \).
  - expect one per horizon volume at \( t \sim 10^{-36} \text{ s} \), i.e. a number density of \( 10^{82} \text{ m}^{-3} \) at \( 10^{-36} \text{ s} \).
  - result: Universe would be today completely dominated by monopoles!
On the Structure Problem

- There is no natural mechanism to produce the fluctuations seen in CMB, in form of Temp flucrts.
All five problems are solved, if the Universe expands very rapidly at some time $t_{\text{inf}}$ where $10^{-38}\text{ s} < t_{\text{inf}} \ll t_{\text{BBN}}$

- Monopole concentration diluted by expansion factor;
- Exponential increase of radius of curvature;
- Visible universe expands from one causally connected region.

this is inflation.

Alan Guth and Andrei Linde, 1981
Inflation and the horizon

- Assume large positive cosmological constant $\Lambda$ acting from $t_{\text{inf}}$ to $t_{\text{end}}$
- then for $t_{\text{inf}} < t < t_{\text{end}}$
  \[ a(t) = a(t_{\text{inf}}) \exp[H_i(t - t_{\text{inf}})] \]
  - $H_i = (\frac{1}{3} \Lambda)^{1/2}$
  - if $\Lambda$ large $a$ can increase by many orders of magnitude in a very short time
- Exponential inflation is the usual assumption but a power law $a = a_{\text{inf}}(t/t_{\text{inf}})^n$ works if $n > 1$
Inflation and Flatness

- We had \( 1 - \Omega(t) = -\frac{kc^2}{H(t)^2 a(t)^2 R_0^2} = \frac{H_0^2(1 - \Omega)}{H(t)^2 a(t)^2} \)
  - for matter-dominated universe \( 1 - \Omega \propto a \)
  - for cosmological constant \( H \) is constant, so \( 1 - \Omega \propto a^{-2} \)

- Assume at start of inflation \( |1 - \Omega| \sim 1 \)
- Now \( |1 - \Omega| \sim 0.06 \)
  - at matter-radiation equality \( |1 - \Omega| \sim 2 \times 10^{-5}, t \sim 50000 \text{ yr} \)
  - at end of inflation \( |1 - \Omega| \sim 10^{-50} \)
  - so need to inflate by \( 10^{25} = e^{58} \)
What powers Inflation?

- We need $H_{\text{inf}}(t_{\text{end}} - t_{\text{inf}}) \geq 58$
  - if $t_{\text{end}} \sim 10^{-34}$ s and $t_{\text{inf}} \sim 10^{-36}$ s, $H_{\text{inf}} \sim 6 \times 10^{35}$ s$^{-1}$
  - this implies $\Lambda \sim 10^{72}$ s$^{-2}$
  - energy density $\varepsilon_{\Lambda} \sim 6 \times 10^{97}$ J m$^{-3}$ ~ $4 \times 10^{104}$ TeV m$^{-3}$
    - cf. current value of $\Lambda \sim 10^{-35}$ s$^{-2}$, $\varepsilon_{\Lambda} \sim 10^{-9}$ J m$^{-3}$ ~ 0.004 TeV m$^{-3}$

- We also need an equation of state with negative pressure

\[
\frac{\dot{a}}{a} = -\frac{4\pi G}{3c^2} (\varepsilon + 3P)
\]
  → accelerating expansion needs $P < 0$

  - cosmological constant $\Lambda$ has $\varepsilon = -P$
Inflation and Particle Physics

At very high energies particle physicists expect that all forces will become unified

- this introduces new particles
- some take the form of scalar fields $\phi$ with equation of state

$$
\varepsilon_{\phi} = \frac{1}{2\hbar c^3} \phi^2 + U(\phi)
$$

$$
P_{\phi} = \frac{1}{2\hbar c^3} \phi^2 - U(\phi)
$$

- if $\phi^2 \ll 2\hbar c^3 U(\phi)$ this looks like...
Inflation with Scalar Field

- Need potential $U$ with broad nearly flat plateau near Higgs field (inflaton field) $\varphi = 0$:
  - metastable false vacuum
  - inflation as $\varphi$ moves very slowly away from 0
  - stops at drop to minimum (true vacuum)
    - decay of inflaton field at this point reheats universe, producing photons, quarks etc. (but not monopoles – too heavy)
    - equivalent to latent heat of a phase transition.
The simplest scenario features a single scalar field moving in a potential $V(\phi)$. Many apparently more complicated scenarios can be reduced to this.

\[
H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \left( V(\phi) + \frac{1}{2} \phi^2 \right)
\]

\[
\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}
\]
The Slow-Roll Approximation

These equations can only be solved exactly for a few choices of potential, for example an exponential potential

\[ V(\phi) \propto \exp(\lambda \phi) ; \lambda^2 < 2 \]

\[ H^2 = \frac{8\pi G}{3} \left( V(\phi) + \frac{1}{2} \dot{\phi}^2 \right) \]

\[ \ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi} \]

However, usually sufficiently accurate results can be obtained by using the slow-roll approximation.

Ordinarily the equations can then be solved analytically. Conveniently, the condition for inflation to occur is almost precisely the same as that for validity of the slow-roll approximation.
EoS for the Inflaton Field

\[ \rho = \frac{1}{2} \dot{\phi}^2 + V(\phi) \]

\[ p = \frac{1}{2} \dot{\phi}^2 - V(\phi). \]

\[ w = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\dot{\phi})} = \frac{1 - (2V/\dot{\phi}^2)}{1 + (2V/\dot{\phi}^2)}, \]

\[ -1 \leq w \leq 1. \]
The Inflaton Field
Slow-Roll Approx

\[ V(\varphi) = \frac{1}{2} m^2 \varphi^2 \Rightarrow V'(\varphi) = m^2 \varphi, \quad V''(\varphi) = m^2 \]

\[ \varepsilon(\varphi) = \frac{1}{2} M_{\text{Pl}}^2 \left( \frac{V'}{V} \right)^2 \]

\[ \eta(\varphi) = M_{\text{Pl}}^2 \frac{V''}{V} \]

\[ \Rightarrow \varepsilon = \eta = 2 \left( \frac{M_{\text{Pl}}}{\varphi} \right)^2 \]

\[ \varepsilon, \eta \ll 1 \Rightarrow \varphi^2 \gg 2 M_{\text{Pl}}^2 \]
Initial Conditions

\[ \therefore \text{The condition for inflation is } \frac{-\dot{H}}{H^2} < 1 \quad (42) \]

If the slow-roll apx is valid,

\[ H^2 = \frac{V}{3M_{\text{Pl}}^2} \quad \Rightarrow \quad 2H\dot{H} = \frac{V'\phi}{3M_{\text{Pl}}^2} \quad \Rightarrow \quad H^2\dot{H} = \frac{V'H\dot{\phi}}{6M_{\text{Pl}}^2} = \frac{3H\dot{\phi} = -V'}{18M_{\text{Pl}}^2} - \frac{V'^2}{18M_{\text{Pl}}^2} \]

\[ \Rightarrow \quad -\frac{\dot{H}}{H^2} = \frac{V'^2}{18M_{\text{Pl}}^2} \frac{9M_{\text{Pl}}^4}{V^2} = \frac{1}{2}M_{\text{Pl}}^2 \left( \frac{V'}{V} \right)^2 = \varepsilon \ll 1 \]

\[ \therefore \text{If the slow-roll apx is valid, inflation is guaranteed.} \]

(sufficient, but not necessary condition)
• A potential $V(\phi)$ must be given, probably from the GUT phase.

• A way to end inflation, e.g. if slow-roll condition is no longer valid.

$x\rightarrow$ Reheating phase, because the inflationary Universe is adiabatically cooling down! Or warm Inflation.

• or when extra physics enters: hybrid inflation.
We can calculate $N(t) \equiv N(\varphi(t)) \equiv N(\varphi)$ from the shape of the potential $V(\varphi)$ and the value of $\varphi$ at time $t$:

$$\dot{a} = Ha \quad \Rightarrow \quad \frac{da}{a} = d \ln a = H dt, \quad \text{where} \quad dt = \frac{d\varphi}{\dot{\varphi}}$$

$$\Rightarrow \quad N(\varphi) \equiv \ln \frac{a(t_{\text{end}})}{a(t)} = \int_{t}^{t_{\text{end}}} H(t) dt = \int_{\varphi}^{\varphi_{\text{end}}} \frac{H}{\dot{\varphi}} d\varphi \quad \text{slow roll} \quad \approx \quad \frac{1}{M_{\text{Pl}}^2} \int_{\varphi_{\text{end}}}^{\varphi} \frac{V}{V'} d\varphi.$$  

(44)
$V = g^4 \left( \phi^4 \ln \phi - \frac{\phi^4}{4} + \frac{1}{4} \right)$
A. Linde: Inflation as Harmonic Oscillator

\[ V(\phi) = \frac{m^2}{2} \phi^2 \]
Coupled Equations of Motion

- Einstein:
  \[ H^2 = \left( \frac{\ddot{a}}{a} \right)^2 = \frac{m^2}{6} \phi^2 \]

- Klein-Gordon:
  \[ \dddot{\phi} + 3H \dot{\phi} = -m^2 \phi \]

Compare with equation for the harmonic oscillator with friction:

\[ \dddot{x} + \alpha \ddot{x} = -kx \]
Logic of Inflation

Large $\Phi$ → large $H$ → large friction

Field $\phi$ moves very slowly, so that its potential energy for a long time remains nearly constant

$$H = \frac{\dot{a}}{a} = \frac{m\phi}{\sqrt{6}} \approx \text{const}$$

$$a \sim e^{Ht}$$

No need for false vacuum, supercooling, phase transitions, etc.
and the field equation can be combined into a closed form for $\Phi$

$$\ddot{\Phi} + \sqrt{\frac{12\pi}{M_P^2}} \dot{\Phi} \sqrt{\dot{\Phi}^2 + m^2 \Phi^2} + m^2 \Phi = 0.$$  (6.42)

This is a nonlinear second order differential equation with no explicit time–dependence. It can be reduced to a first order differential equation by using the relation

$$\ddot{\Phi} = \dot{\Phi} \frac{d\dot{\Phi}}{d\Phi}$$  (6.43)

with the result

$$\frac{d\dot{\Phi}}{d\Phi} = -\frac{\sqrt{\frac{12\pi}{M_P^2}} \sqrt{\dot{\Phi}^2 + m^2 \Phi^2} \dot{\Phi} + m^2 \Phi}{\dot{\Phi}}.$$  (6.44)

This can be studied using phase diagram methods. The behaviour of the solutions is shown in the $\Phi - \dot{\Phi}$ plane in Fig. 6.11. The important feature of this diagram is the existence of an attractor solution to which all other solutions converge in time. We restrict our discussions to the lower right quadrant with positive values for $\Phi$ and $\dot{\Phi} < 0$. 
Attractors
Oscillations around Minimum

Initial Condition

Phase Plot for Chaotic Inflation

Oscillations around Minimum

Attractors
Inflation makes the Universe flat, homogeneous and isotropic

In this simple model the universe typically grows $10^{30}$ times during inflation.

Now we can see just a tiny part of the universe of size $ct = 10^{10}$ light yrs. That is why the universe looks homogeneous, isotropic, and flat.
Hybrid Inflation: 2 Fields

\[ V(\sigma, \phi) = \frac{1}{4\lambda} (M^2 - \lambda \sigma^2)^2 + \frac{m^2}{2} \phi^2 + \frac{g^2}{2} \phi^2 \sigma^2 \]
String Theory Landscape

Perhaps $10^{100} - 10^{1000}$ different minima

Lerche, Lust, Schellekens 1987

Bousso, Polchinski; Susskind; Douglas, Denef, ...
Example: Racetrack Inflation

waterfall from the saddle point
**Example: SuSy Landscape**

Supersymmetric SU(5)

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**Weinberg 1982**: Supersymmetry forbids tunneling from SU(5) to SU(3)xSU(2)xU(1). This implied that we cannot break SU(5) symmetry.

**A.L. 1983**: Inflation solves this problem. Inflationary fluctuations bring us to each of the three minima. Inflation make each of the parts of the universe exponentially big. We can live only in the SU(3)xSU(2)xU(1) minimum.
Self-Reproducing Inflationary Universe

A. Linde
Uncertainty Principle means that in quantum mechanics vacuum constantly produces temporary particle-antiparticle pairs:

- minute density fluctuations;
- inflation blows these up to macroscopic size;
- seeds for structure formation;

Expect spectrum of fluctuations to be approximately scale invariant

possible test of inflation idea?
What was the physical size of cosmological scales contributing to the CMB today before inflation?

This depends on the number of e-folds of inflation. Most models give more than the minimum of 60’ish e-folds.

Generically, those scales begin at sizes less than the Planck scale! Certainly, we should expect these scales to encompass new physics thresholds.

Does new physics stretch as well?
Linear theory (coordinate approach)

- Perturbed Friedmann universe

\[ ds^2 = -(1 + 2A) c^2 dt^2 + a^2(t) (1 - 2\Psi) \delta_{ij} dx^i dx^j \]

- proper time along \( x^i = \text{const.} \):
  \[ d\tau = (1 + A) dt \]

- curvature perturbation on \( \Sigma(t) \):
  \( \psi \leftrightarrow (3) \frac{4}{a^2} \Delta \psi \)
Quantum Fluctuations in $\Phi$

Inhomogeneous spacetime in Newtonian gauge (vanishing stress) (see next Sect. on Perturbations)

\[ ds^2 = a^2(\eta) \left[ -(1 + 2\Phi)d\eta^2 + (1 - 2\Phi)\gamma_{ij}dx^idx^j \right]. \]

Einstein's equations imply ~ Klein-Gordon equation with mass-term:

\[ \Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' - c_s^2\Delta^{(3)}\Phi + [2\mathcal{H}' + (1 + 3c_s^2)(\mathcal{H}^2 - \mathcal{K})]\Phi = 4\pi G a^2 \tau \delta S. \]

Adiabatic fluctuations $\Rightarrow dS = 0$
\( \rightarrow \text{Wave Equation in } \Phi \sim u \)

Make a rescaling, such that 1st order derivative disappears:

\[
\begin{align*}
  u'' - c_s^2 \Delta^{(3)} u - \frac{z''}{z} u &= 0, \\
  z &= \frac{a}{c_s H} \left( \mathcal{H}^2 - \mathcal{H}' + \mathcal{K} \right)^{1/2}. 
\end{align*}
\]

\( \rightarrow \text{Klein-Gordon equation with mass given by } m^2 = -z''/z \text{ (Barrier)} \)
\( \rightarrow \text{Can be quantized similar} \)

For \( K = 0 \) and \( c_s = c \):

\( \rightarrow \text{Inflaton field is the source for metric perturbations.} \)
Quantization of $\Phi$

\[ \int d^4x \sqrt{\gamma} \mathcal{L} = \frac{1}{2} \int d^4x \sqrt{\gamma} \left( u'^2 - \gamma^{ij} u_i u_j + \frac{z''}{z} u^2 \right) , \]

\[ \hat{u}(\eta, x) = \int \frac{d^3p}{\sqrt{2\pi}^3} \left[ \hat{a}_k u_k(\eta) \exp(ik \cdot x) + \hat{a}^\dagger_k u_k^*(\eta) \exp(-ik \cdot x) \right] . \]

\[ u''_k(\eta) + \left[ k^2 - \frac{z''(\eta)}{z(\eta)} \right] u_k(\eta) = 0. \]

Short wavelength limit

\[ u_k \rightarrow \frac{1}{\sqrt{2k}} e^{-i k \eta} \quad (k^2 \gg |z''/z| , -k\eta \rightarrow \infty) , \]

\[ u_k \rightarrow A_k z \quad (k^2 \ll |z''/z| , -k\eta \rightarrow 0) . \]

Long wavelength limit
The Mode Solution

Inside Horizon

Mode freezing

Mode leaving the Horizon is frozen in.

Outside Horizon

\[ u''_k + \left( k^2 - \frac{a''}{a} \right) u_k = 0, \]

\[ k = \frac{2\pi}{\lambda} \]

Simple Scalar Field in a de Sitter Inflation

\[ \log(ck\eta) \]

\[ \frac{u_k}{a} \]
Potential Barrier and Power Spec

Power Spectrum

\[ P_S(k) = \lim_{k \eta \to 0^-} \frac{k^3}{2\pi^2} \left| \frac{u_k(\eta)}{z(\eta)} \right|^2, \]

\[ P_T(k) = \lim_{k \eta \to 0^-} \frac{k^3}{2\pi^2} \left| \frac{v_k(\eta)}{a(\eta)} \right|^2, \]

\[ \mathcal{P}_R = \frac{k^3}{2\pi^2} \langle |\mathcal{R}|^2 \rangle = \frac{k^3}{2\pi^2} \frac{H^2}{\phi_0^2} |\delta \phi_k|^2. \]
Conventional Slow-Roll Approx

\[ \epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{1}{2} \left( \frac{\dot{\phi}}{H} \right)^2, \quad \delta_n \equiv \frac{1}{H^{n-1}} \frac{d^{n+1}\phi}{dt^{n+1}}. \]

\[ \frac{z''}{z} = 2a^2H^2 \left( 1 + \epsilon + \frac{3}{2} \delta_1 + 2\epsilon \delta_1 + \epsilon^2 + \frac{1}{2} \delta_2 \right). \]

\[ u''_k(\eta) + \left( k^2 - \frac{A}{\eta^2} \right) u_k(\eta) = 0, \]

\[ A = 2 + 6\epsilon + 3\delta_1 = \text{constant}. \]

\[ \rightarrow \text{Solution by Hankel functions} \]
• Parameters:
  • Chaotic Inflation
  • $m^2 = 1.9 \times 10^{-12} \, M_P^2$
  • $\Phi(0) = 16.8 \, M_P$
  • $d\Phi(0) = -0.1 \, M_P/s$

$N = 57.65$

$M_P^2 = 1/8\pi G$

Andreas Heinen
Thesis 2005
Power Spectrum – Grav. Waves

Andreas Heinen
Thesis 2005
Power Spectrum - Running Spectral Index

Running spectral index:

\[ n_s(k) = 1 + \frac{d \ln P_s(k)}{d \ln k}, \]
\[ n_T(k) = \frac{d \ln P_T(k)}{d \ln k}. \]
Spectral Index – Grav. Waves

Andreas Heinen
Thesis 2005
Ratio $R = \text{Grav. Waves} / \text{Scalar}$
• **Parameters:**
  
  - **Quartic Inflation:** $\sim \lambda \Phi^4$
  
  $\forall \lambda = 1.75 \times 10^{-13}$
  
  $\Phi(0) = 24 \, M_P$
  
  $d\Phi(0) = -1 \, M_P/s$

  $\Rightarrow N = 60.58$

  $M_P^2 = 1/8\pi G$
While the Universe is inflating, its contents is cold. But eventually, inflation has to end and the field driving the inflation must decay, depositing energy into high-energy particles. This process, known as reheating, “boils” the vacuum and starts the thermal history of the Universe with the hot Big Bang. As the Universe continues to cool down, it could undergo more phase transitions, which would happen at symmetry breaking points of the theory. Very little is known about the fundamental physics at these energy scales, and cosmological observations could be our only source of information for the foreseeable future. No photons reach us directly from this epoch, as the Universe is filled with hot plasma and is opaque until recombination.
Reheating the Universe

\[ V(\phi, \chi) = \frac{1}{4} \lambda \phi^4 + \frac{1}{2} g \phi^2 \chi^2. \]
Horizon size (red):  
(i) $\sim$ constant during Inflation;  
(ii) turns at end of Inflation to radiation-dominance;  
(iii) $d_H = 2ct \sim a^2$ in rad-dominance.  

$\Rightarrow$ Perturbations leave horizon and re-enter later on.

DEFROST: Frolov 2010
Inflationary Universe

Quantum Fluctuations in Potential $\Phi$ generated in Inflation

Wavelengths are simply stretched in Expansion

Das Inflationäre Universum
Fluctuations in Density / Sim

Inflaton mass $m = 5 \times 10^{-6} M_{Pl} \sim 10^{13}$ GeV/c²

Box size: $L = 10/m$

DEFROST: Frolov 2008
Fluctuations in $\Phi$
Inflation – Connection to Quantum Gravity?
Inflation scenario predicts:
- Universe should be very close to flat;
- No causality problem;
- CMB should be isotropic, with small scale invariant perturbations – seeds for cosmic web and galaxies;
- Monopole number density unobservably low.

Inflation scenario does not predict:
- current near-equality of $\Omega_m$ and $\Omega_{\Lambda}$
- matter-antimatter asymmetry.

Underlying particle physics very difficult to test
- energy scale is much too high for accelerators!
Inflation solves Flatness problem, Horizon problem & many other aspects: N > 55.

Inflation also provides source for perturbations on the Friedmann backgrd. by means of quantum fluctuations in the very Early Universe $\Phi \sim 10^{-5}$.

These perturbations are frozen in, once they are stretched by expansion beyond the horizon. Later on, they will enter the horizon.

Power spectrum and spectral index will depend on inflation model, but $n_s \sim 0.962$. 