Albert Einstein
German
General Theory of Relativity (1915);
Static, closed universe (1917)

W. de Sitter
Dutch
Vacuum-energy-filled universes
"de Sitter space" (1917)

H.P. Robertson
American
Formalized most general form of isotropic
and homogeneous universe in GR
"Robertson-Walker metric" (1935-6)

A.G. Walker
British

A. Friedmann
Russian
Evolution of homogeneous, non-static (expanding) universes
"Friedmann models" (1922, 1927)

G. LeMaitre
Belgian

"burst of fireworks" 1927
Invents the Big Bang
Basic Problem: How to describe the expanding Universe? – only via Einstein

The Cosmological Principle

Metric of SpaceTime

FRW Models

Origin of Redshift

The Hubble-Law as first order expansion

The luminosity distance

Angular distance

Number counts in expanding Universe
Cosmic Microwave Background

- “Afterglow” of the big-bang
- Discovered by Penzias and Wilson (1965)

→ glimpse of the universe in its infancy

<\( T \) = 2.73°K

Small temperature fluctuations arising from early weak fluctuations in the matter distribution
The Cosmological Principle ➔ the Universe is homogeneous and isotropic on sufficiently large scales

The universe looks pretty much like this everywhere – “walls” and “voids” are present but no larger structures are seen….

It follows that the Universe has no “edge” or center.

But is the Universe the same at all times?
• How can we synchronize this Universal cosmological time coordinate, $t$, everywhere?

• With a Symmetry Principle:
  • On large-scales Universe seems isotropic (same in all directions, eg, Hubble expansion, galaxy distribution, CMB).
  • Combine with Copernican Principle (we’re not in a special place).
Relativistic Cosmologies

- Isotropy + Copernican Principle = homogeneity (same in all places)

So $\rho_2 = \rho_1 = \rho_0$.

So uniform density everywhere
What is the form of space, $ds_3^2$?

The Cosmological Principle (Homogeneity + Isotropy) $\rightarrow$ only constant Curvature allowed.

$\rightarrow$ Only 3 Possibilities:
- 3-Sphere – positive Curvature $K > 0$
- 3-Saddle – negative Curvature $K < 0$
- Flat $\mathbb{E}^3$ – no Curvature $K = 0$
Metric of a 2-Sphere

- The metric on a 2-sphere of radius $R$:
  \[
  d\sigma_2^2 = R^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)
  \]

- Now re-label $\theta$ as $r$ and $\phi$ as $\theta$:
  \[
  d\sigma_2^2 = R^2 (dr^2 + \sin^2 r \, d\theta^2)
  \]

where $r = (0, \pi)$ is a dimensionless distance.
The Robertson-Walker models:

\[ c^2 d\tau^2 = c^2 dt^2 - R^2 (dr^2 + S_k^2(r) d\psi^2) \]

- \( k = +1 \): positive curvature everywhere, spatially closed, finite volume, unbounded.
- \( k = -1 \): negative curvature everywhere, spatially open, infinite volume, unbounded.
- \( k = 0 \): flat space, spatially open, infinite volume, unbounded.
SpaceTime View of the Expansion of the Universe

3-space is expanding: $dx \rightarrow R(t) \, dx$
Wavelengths also are stretched.
A Simple Model: Flat Space

The simplest ansatz: expanding Euclidean Space: $dx \rightarrow R(t) \, dx$ in all directions.

Standard 3-SpaceMetric:

\[ ds^2 = dx^2 + dy^2 + dz^2 \]
\[ = dr^2 + r^2d\theta^2 + r^2\sin^2\theta \, d\phi^2 \]

In Cartesian or Spherical coordinates in Euclidean Space.
Now make our space expanding, but “homogeneous” & “isotropic” →

\[ ds^2 = R^2(t) \left( dx^2 + dy^2 + dz^2 \right) \]

And then allow transformation to a more general geometry (i.e. allow non-Euclidean geometry) but keep isotropic and homogeneous:

\[ ds^2 = c^2dt^2 - R^2(t) \left( dx^2 + dy^2 + dz^2 \right) \]
Spaces of constant curvature are conformally flat:

\[ ds^2 = (1+1/4kr^2)^2 \left( dx^2 + dy^2 + dz^2 \right) R^2(t) \]

where \( r^2 = x^2 + y^2 + z^2 \), and \( k \) is a measure of space curvature.

Note the Special Relativistic Minkowski Metric

\[ ds^2 = c^2 dt^2 - \left( dx^2 + dy^2 + dz^2 \right) \]
So, if we take our general metric and add the 4\textsuperscript{th} (time) dimension, in conformal coords:

\[ ds^2 = c^2 dt^2 - R^2(t)(dx^2 + dy^2 + dz^2)/(1+kr^2/4) \]

or in spherical coordinates and simplifying,

\[ ds^2 = c^2 dt^2 - R^2(t)[dr^2/(1-kr^2) + r^2(d\theta^2+\sin^2\theta \ d\phi^2)] \]

which is the (Friedmann)-Robertson-Walker Metric \( \Rightarrow \) FRW model.
• The FRW metric is the most general, non-static, homogeneous and isotropic metric. It was derived ~1930 by Robertson and Walker.

\[ R(t), \text{ the Scale Factor, is an unspecified function of time (which is usually assumed to be continuous)} \]

\[ \text{and } k = 1, 0, \text{ or } -1 = \text{ the Curvature Constant.} \]

\[ \text{For } k = -1 \text{ or } 0, \text{ space is infinite.} \]
Mathematical Intermezzo

CMB $\rightarrow$ Family of preferred world lines, representing average motion of matter

Isotropy around each point $\rightarrow$ homogeneous $\rightarrow$ 3 space is a space of constant curvature $K$

Dim $> 2$ $\rightarrow$ spaces of constant curvature are conformally flat

$$
\gamma = \sum_{i=1,n} \frac{(dx^i)^2}{(1 + K\rho^2/4)^2}, \quad \rho^2 = \sum_i (x^i)^2.
$$

$$
r = \frac{\rho}{1 + K\rho^2/4}, \quad dr = \frac{r}{\rho} d\rho - \frac{K}{2} r^2 d\rho
$$
r, θ, φ are co-moving Coordinates (“Labels” for Objects).

t: cosmological time (measured by atomic clocks in the Center of Galaxy Clusters).

\[ dx = a(t) \, dr : \text{Distances are stretched (isotropic Expansion).} \]

\[ a(t) \text{ is a Function of time and } r \text{ remains constant.} \]

\[ a(t) \text{ is the Scale-factor of the Universe and measures the universal expansion rate of the Universe.} \]

\[ a(t_0) = 1, \text{ where } t_0 \text{ is the present cosmological time.} \]
FRW Model of the Universe

- in quasi–spherical coordinates \((t, r, \theta, \phi)\)

\[
ds^2 = c^2 \, dt^2 - R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right].
\]

- in conformal coordinates \((t, \rho, \theta, \phi)\)

Riemann spaces of constant curvature are conformally flat

\[
ds^2 = c^2 \, dt^2 - \frac{R^2(t)}{(1 + k\rho^2/4)^2} \left[ d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right].
\]

- in hyper–spherical coordinates \((t, \chi, \theta, \phi)\)

\(r = 1 \ (k = +1)\) is not singular

\[
ds^2 = c^2 \, dt^2 - R^2 \left[ d\chi^2 + S^2(\chi) (d\theta^2 + \sin^2 \theta \, d\phi^2) \right],
\]

where \(S(\chi) = \{\sin \chi, \chi, \sinh(\chi)\}\) depending on the curvature.
Is FRW Cosmology a good approximation??

YES!

YES!

YES!
Physical Effects in FRW Models

- Light propagates along Null geodesics.
- Expansion $\Rightarrow$ cosmological redshift.
  $\Rightarrow$ explains the Hubble law.
- Distance is not distance!
  $\Rightarrow$ Proper distance.
  $\Rightarrow$ Luminosity distance.
  $\Rightarrow$ Angular distances.
  $\Rightarrow$ Observed volume
SpaceTime is Filled by Worldlines of Galaxies
1. Light Propagation: along Null-Geodesics

- How do photons propagate in the expanding Universe?
- Consider a photon emitted at \((r_e)\) along a Line with const Longitude & Latitude \((d\theta = 0 = d\phi)\).
- The Trajectory is a Null-Geodesic \((\text{Eigenzeit} = 0)\).

\[
c^2 d\tau^2 = c^2 dt^2 - R^2(t) dr^2 = 0 \quad k = 0
\]
Light Propagation under

- Equation of motion for Photons ($a = R$):

\[ c^2 dt^2 = R^2(t) dr^2 \]

\[ r(t) = \int_0^t \frac{cdt}{R(t)} \]

“Comoving distance”

\( \Rightarrow \) comoving Distance decreases.
2. Cosmological Redshift

\[ \int_{t_e}^{t_0} \frac{c \, dt}{R(t)} = \int_0^{r_e} \frac{dr}{\sqrt{1 - kr^2}}. \]

\[ \int_{t_e + \Delta t_e}^{t_0 + \Delta t_0} \frac{c \, dt}{R(t)} = \int_0^{r_e} \frac{dr}{\sqrt{1 - kr^2}}. \]
RHS identical \(\Rightarrow\)

\[
\int_{t_e+\Delta t_e}^{t_0} \frac{c \, dt}{R(t)} + \int_{t_0}^{t_0+\Delta t_0} \frac{c \, dt}{R(t)} = \int_{t_e}^{t_e+\Delta t_e} \frac{c \, dt}{R(t)} + \int_{t_e+\Delta t_e}^{t_0} \frac{c \, dt}{R(t)}.
\]

The first term cancels the last one \(\Rightarrow\)

\[
\frac{c \Delta t_0}{R(t_0)} = \frac{c \Delta t_e}{R(t_e)}.
\]

\[
\frac{c \Delta t_0}{c \Delta t_e} = \frac{\nu_e}{\nu_0} = \frac{\lambda_0}{\lambda_e} = \frac{R(t_0)}{R(t_e)} = 1 + z,
\]

Wavelengths are stretched by expansion!
cosmological redshift \[
z = (\lambda_o - \lambda_e) / \lambda_e\]

scale factor: \[
a(t_e)v_e = a(t_o)v_o\]

\[
a = 1/(1+z)\]
Redshift, Photon Cooling and the CMB

Redshift law in an expanding Universe:

\[ \lambda (t) = \lambda_{em} \times \left( \frac{R(t)}{R_{em}} \right)^{-1}, \quad z = \left( \frac{R(t)}{R_{em}} \right) - 1 \]

Photon energy (Planck, 1900):

\[ E = h\nu = hc / \lambda \]

Photons cool in an expanding Universe:

\[ E(t) = E_{em} \times \left( \frac{R(t)}{R_{em}} \right)^{-1} \]
3. The Luminosity Distance

\[ F \Delta t_0 = \frac{L \Delta t_1}{4\pi r_1^2 R^2(t_0)} \frac{R(t_1)}{R(t_0)}, \]

i.e. the observed flux is given by

\[ F = \frac{\Delta t_1}{\Delta t_0} \frac{R(t_1)}{R(t_0)} \frac{L}{4\pi R^2(t_0)r_1^2} = \frac{L}{4\pi R_0^2 r_1^2} \left( \frac{R(t_1)}{R(t_0)} \right)^2. \]

**Luminosity Distance:**

\[ d_L = r_e R(t_0)(1 + z) \]
Luminosity Dist and Hubble-Law

\[ R(t) = R(t_0) + \dot{R}(t - t_0) + \frac{1}{2} \ddot{R}(t - t_0)^2 + O(\Delta t^3) \]

\[ = R_0 \left( 1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + O(\Delta t^3) \right). \]

\[ H_0 = \frac{\dot{R}}{R_0} \]

\[ q_0 = -\frac{\ddot{R}}{(R_0 H_0^2)}. \]

\[ z = H_0(t_0 - t) + \left( 1 + \frac{1}{2} q_0 \right) H_0^2 (t - t_0)^2. \]

\[ 1 + z = \left( 1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + O(\Delta t^3) \right)^{-1} \]

\[ \approx 1 + H_0(t_0 - t) + \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + H_0^2 (t - t_0)^2 + O(\Delta t^3). \]
For not too early times, i.e. $H_0|t - t_0| \ll 1$, this can be inverted to give

$$t_0 - t = \frac{1}{H_0} \left( z - \left(1 + \frac{q_0}{2}\right) z^2 \right) + O(z^3).$$

For $r R_0$:

$$r R_0 = \int_t^{t_0} \frac{R_0}{R(t)} \, dt = \frac{1}{R_0} \int_t^{t_0} (1 + z) \, dt$$

$$= \frac{1}{R_0} \int_t^{t_0} \left[ 1 + H_0 \tau + \left(1 + \frac{1}{2} q_0\right) H_0^2 \tau^2 \right] \, d\tau$$

$$= c(t_0 - t) \left( 1 + \frac{H_0}{2} (t_0 - t) + O([H_0(t - t_0)]^2) \right).$$

For $r R_0$:

$$r R_0 = \frac{c}{H_0} \left( z - \left(1 + \frac{q_0}{2}\right) z^2 + \frac{1}{2} z^2 + O(z^3) \right)$$

$$= \frac{c}{H_0} \left( z - \frac{1}{2} (1 + q_0) z^2 + O(z^3) \right).$$
Hubble constant is just the expansion velocity of the present Universe.
Angular Distance $D_A$

$$D_A = \frac{l}{\delta \theta}$$

$$dS^2 = c^2 d\tau^2 - R^2(t) \left( \frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right)$$

$$l = R(t_e)r \delta \theta = \frac{R_o r \delta \theta}{1+z}$$

$$D_A = \frac{l}{\delta \theta} = \frac{R_o r}{1+z}$$

In source system:

$$dS^2 = l^2 = R(t_e)^2 r^2 \delta \theta$$

$R(t_e)$ at the source
Cosmic Angular Diameter

- Defined as $\theta = D / d_A = D (1+z)^2 / d_L$
  
  - $D =$ physical Dimension of an Object
  - $\theta =$ Angular Diameter on Sky

\[
d_A = \frac{d_L}{(1+z)^2} = \frac{D_C(z)}{1+z}
\]

$\Rightarrow$ Apparent angular diameter has Minimum at $z \sim 1.4$ !!!!
Surface Brightness

Surface brightness is flux per unit solid angle

\[ B = \frac{f}{d\omega} \]

\[ B = \frac{L}{D_L^2} \frac{D_A^2}{dl^2} \]

\[ B = \frac{L}{dl^2} \frac{D_A^2}{D_L^2} = \frac{L}{dl^2} (1 + z)^{-4} \]

Independent of cosmology!
Number Counts in the Universe

COSMOS 1.4Msec 2deg$^2$

Lockman Hole 0.7Msec 0.3deg$^2$
C(q): point q of observation along C (observer’s world line)

\[ dN = d_A^2 d\Omega_0 (1 + z) n(y) dy. \]
Number Counts in Universe

\[ dl = c \, dt = c \frac{dt}{dR} \frac{dR}{dz} \, dz = - \frac{dz}{1 + z} \frac{cR}{\dot{R}} = - \frac{dz}{1 + z} \, d_H(z), \]

\[ dV = \frac{d_H(z) \, dz}{1 + z} d_A^2 \, d\Omega = \frac{R_0^2 r_{em}^2(z) d_H(z)}{(1 + z)^3} \, dz \, d\Omega. \]

\[ \frac{dN}{dz \, d\Omega} = n(z) \frac{dV}{dz \, d\Omega} = n(z) \frac{R_0^2 r_{em}^2(z) d_H(z)}{(1 + z)^3} = n(z) \, d_H(z) \frac{d_L^2(z)}{(1 + z)^5}. \]
Expected SNAP cluster counts

15 square degree survey still suffers from cosmic variance. 300 square degree lensing survey will be more useful for cluster counts.

Based on Virgo consortium Hubble Volume ΛCDM Simulations

Redshift in bins of 0.02

\[ N(M > 5 \times 10^{13} M_\odot) \text{ in 15 sq. deg.} \]
Curved Spacetime in a Closed Universe

\[ N \text{ galaxies in } x \text{ direction } \Rightarrow D(t) = \frac{2\pi R(t)}{N} \]
Summary

• Friedmann Universe is given by 3-spaces of constant curvature which are expanding with time

\[ a(t) \]

metric with only one degree of freedom: scale factor \( a(t) \)

+ curvature \( k = 0,+-1 \) of 3-space.

redshift \( 1+z \) is measure of expansion, \( 1 + z = 1/a(t), a(t_0) = 1. \)

In lowest order, Hubble-law is satisfied, quadratic correction given by deceleration \( q_0 \).