Modern Cosmology
Exercises 3: Friedmann Models

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1. Spaces of Constant Curvature – FRW Models

The metric for an isotropic 3–space has the ansatz of the form, as used e.g. in the derivation of the Schwarzschild metric,

$$\boxed{d\sigma^2_{(3)} = B(r)\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2\,.} \tag{1}$$

The corresponding non–vanishing Ricci–tensors can be derived to be

$$R_{rr} = \frac{1}{rB}\frac{dB}{dr} \quad , \quad R_{\theta\theta} = 1 + \frac{r}{2B^2}\frac{dB}{dr} - \frac{1}{B} \quad , \quad R_{\phi\phi} = R_{\theta\theta}\, \sin^2\theta\,. \tag{2}$$

This is a **maximally symmetric 3–space**, if

$$\boxed{R_{ijk\ell} = K\left[g_{ik}g_{j\ell} - g_{i\ell}g_{jk}\right],} \tag{3}$$

with some constant K, or for the Ricci tensor $R_{ik} = R^m_{\ imk}$

$$R_{ik} = 2K\, g_{ik}\,. \tag{4}$$

This condition for the Ricci tensor leads to two equations

$$\frac{1}{rB}\frac{dB}{dr} \;=\; 2K\, B(r) \tag{5}$$

$$1 + \frac{r}{2B^2}\frac{dB}{dr} - \frac{1}{B} \;=\; 2Kr^2\,. \tag{6}$$

Show that the solution of the first equation is

$$B(r) = \frac{1}{A - Kr^2}\,, \tag{7}$$

while the second equation gives

$$1 - A + Kr^2 = Kr^2 \quad , \quad \text{i.e.} \quad A = 1\,. \tag{8}$$

In this way, we have derived the general form of the metric for a constant curvature 3–space

$$\boxed{d\sigma^2_{(3)} = \frac{dr^2}{1 - Kr^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2\,,} \tag{9}$$

which is the basis for the spacetime metric of the FRW model $ds^2 = -dt^2 + a^2(t)\, d\sigma^2_{(3)}$.

- Derive the **metric of a 3–sphere** of radius a embedded into a 4–dimensional Euclidean space by means of hyperspherical coordinates χ, θ, ϕ.
- What is the corresponding volume element?
- Derive the **metric of a 3–hyperboloid** of radius a embedded into a 4–dimensional Monkowski space by means of hyperspherical coordinates χ, θ, ϕ.
- Give the metric of the resulting FLRW Universe in hyperspherical coordinates for all 3 cases of space sections: 3–sphere, Euclidean flat space and 3–hyperboloid.

2. Einstein's Equations

- Write down the Riemann tensor in coordinates for given christoffel symbols, the Ricci tensor, the Ricci scalar and the Einstein tensor.
- Write down Einstein's field equations including the cosmological constant.
  What is the meaning of the cosmological constant?
- Give expressions for the Planck mass, Planck length, Planck time and Planck temperature with their numerical values.
- Why is Einstein's theory not globally correct?
- What is the Brans–Dicke theory of gravity?
- What is a metric theory of gravity?

3. Friedmann Equations

- Write down the two Friedmann equations for the expansion factor $R(t)$.
  What is the meaning of these two equations?
- Define the **Hubble–parameter** $H(t)$ and the cosmological Ω–parameters in an expanding universe.
- Give the **density evolution** for DM, photons and vacuum energy in terms of the redshift z.
- Evaluate the energy density ρ_r for the relativistic particles at present time (photons and 3 types of neutrinos).
- How can one determine ρ_m for the present Universe?
- What is the de **Sitter solution**? Give $a(t)$.
- What is the **Einstein–de-Sitter solution**? Give $a(t)$.
- Discuss the solutions for a **Universe without matter**, $\rho_m = 0 = \Lambda$, also known as the coasting Universe.
  Derive the luminosity distance for this model.
- Derive the luminosity distance $d_L(z)$ for a deSitter Universe.