Modern Cosmology
Solutions 3: GR and Friedmann Cosmos

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1. Spaces of Constant Curvature

- In differential geometry, constant curvature is a concept most commonly applied to surfaces. For those the scalar curvature is a single number determining the local geometry, and its constancy has the obvious meaning that it is the same at all points. The circle has constant curvature, also, in a natural (but different) sense.

  For higher dimensional manifolds, constant curvature is usually taken to mean constant sectional curvature, and a complete manifold of this kind is called a space form. As in the case of surfaces, there are three types of geometries (elliptic, flat, or hyperbolic) according to whether the curvature is positive, zero, or negative. The universal cover of a manifold of constant sectional curvature is one of the model spaces (sphere, Euclidean space, hyperbolic space).

  Spaces of **constant curvature** satisfy for the Riemann tensor

  $$\boxed{R_{ijk\ell} = K\left[g_{ik}g_{j\ell} - g_{i\ell}g_{jk}\right],} \tag{1}$$

  with some constant K, or for the Ricci tensor $R_{ik} = R^m_{\ imk}$

  $$R_{ik} = 2K\, g_{ik}\,. \tag{2}$$

  For Dim = 3, there are only 3 types of spaces of constant curvature: 3–sphere ($K > 0$), flat Euclidean space ($K = 0$) and a hyperboloid ($K < 0$).

- **The 3–Sphere:** A 3-sphere is a higher-dimensional analogue of a 2-sphere. It consists of the set of points equidistant from a fixed central point in 4-dimensional Euclidean space. Just as an ordinary sphere (or 2-sphere) is a two dimensional surface that forms the boundary of a ball in three dimensions, a 3-sphere is an object with three dimensions that forms the boundary of a ball in four dimensions. A 3-sphere is an example of a 3-manifold.

  It is convenient to have some sort of hyperspherical coordinates on S^3 in analogy to the usual spherical coordinates on S^2. One such choice – by no means unique – is to use (χ, θ, ϕ), where the embedding into the Euclidean 4–space

  $$x_0^2 + x_1^2 + x_2^2 + x_3^2 = a^2 \tag{3}$$

is given by

$$
\begin{aligned}
x_0 &= a\cos\chi && (4)\\
x_1 &= a\sin\chi\cos\theta && (5)\\
x_2 &= a\sin\chi\sin\theta\cos\phi && (6)\\
x_3 &= a\sin\chi\sin\theta\sin\phi\,. && (7)
\end{aligned}
$$

The angles χ and θ run over the range 0 to π, and ϕ runs over 0 to 2π. The differentials in terms of the polar angles are

$$
\begin{aligned}
dx_0 &= -a\sin\chi\,d\chi && (8)\\
dx_1 &= a\cos\chi\cos\theta\,d\chi - a\sin\chi\sin\theta\,d\theta && (9)\\
dx_2 &= a\cos\chi\sin\theta\cos\phi\,d\chi + a\sin\chi\cos\theta\cos\phi\,d\theta - a\sin\chi\sin\theta\sin\phi\,d\phi && (10)\\
dx_3 &= a\cos\chi\sin\theta\sin\phi\,d\chi + a\sin\chi\cos\theta\sin\phi\,d\theta + a\sin\chi\sin\theta\cos\phi\,d\phi\,. && (11)
\end{aligned}
$$

At the same time we have from equ (3)

$$2x_0\,dx_0 + 2x_1\,dx_1 + 2x_2\,dx_2 + 2x_3\,dx_3 = 0\,, \tag{12}$$

which allows us to express dx_0 in terms of dx_1, dx_2 and dx_3. The metric on the 3-sphere in these coordinates is given by

$$
\begin{aligned}
d\sigma^2 &= dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2\\
&= a^2\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] && (13)
\end{aligned}
$$

and the volume form by

$$dV = a^3\,\left(\sin^2\chi\,\sin\theta\right)\,d\chi\wedge d\theta\wedge d\phi\,. \tag{14}$$

- **The 3–Hyperboloid:** Consider the equation of a 3-hyperboloid embedded in a 4-dimensional Minkowski space

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = a^2 \tag{15}$$

in the sense that

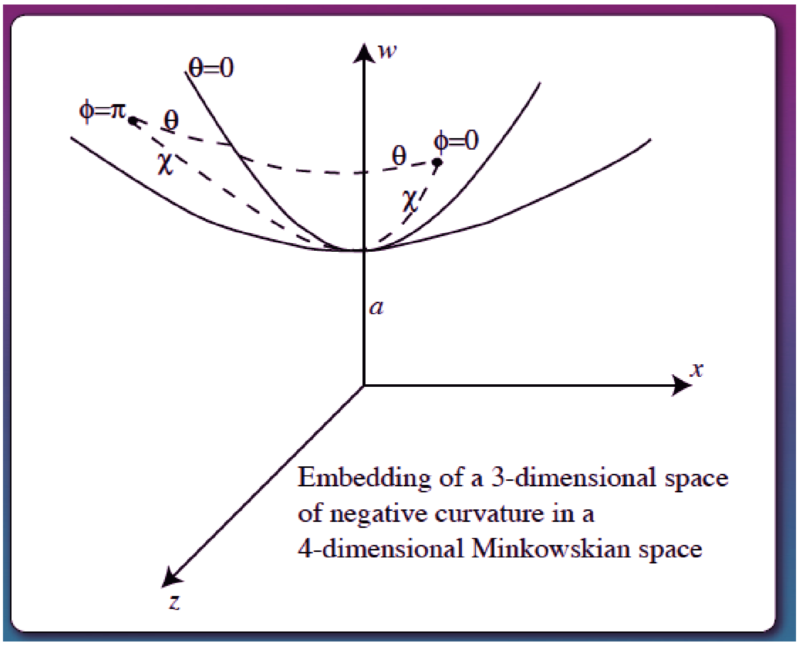
$$
\begin{aligned}
x_0 &= a\cosh\chi && (16)\\
x_1 &= a\sinh\chi\cos\theta && (17)\\
x_2 &= a\sinh\chi\sin\theta\cos\phi && (18)\\
x_3 &= a\sinh\chi\sin\theta\sin\phi\,. && (19)
\end{aligned}
$$

The angle χ runs over $-\infty$ to ∞, θ runs over the range 0 to π, and ϕ runs over 0 to 2π.
In the same procedure as above one can calculate the differentials

$$
\begin{aligned}
dx_0 &= a\sinh\chi\,d\chi && (20)\\
dx_1 &= a\cosh\chi\cos\theta\,d\chi - a\sinh\chi\sin\theta\,d\theta && (21)\\
dx_2 &= a\cosh\chi\sin\theta\cos\phi\,d\chi + a\sinh\chi\cos\theta\cos\phi\,d\theta - a\sinh\chi\sin\theta\sin\phi\,d\phi && (22)\\
dx_3 &= a\cosh\chi\sin\theta\sin\phi\,d\chi + a\sinh\chi\cos\theta\sin\phi\,d\theta + a\sinh\chi\sin\theta\cos\phi\,d\phi && (23)
\end{aligned}
$$

replace dx_0 in terms of the others, and in this way deriving the metric on the 3-hyperboloid in these coordinates

$$\begin{aligned} d\sigma^2 &= dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \\ &= a^2 \Big[d\chi^2 + \sinh^2\chi \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \Big] . \end{aligned} \tag{24}$$



- **The metric of the resulting spacetime** is then given by

$$ds^2 = -c^2\, dt^2 + a^2(t) \Big[d\chi^2 + \mathcal{S}(\chi)^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \Big] . \tag{25}$$

The function $\mathcal{S} = (\sin\chi, \chi, \sinh\chi)$, depending on the curvature of the space section. This is one of 3 forms for the metric of the FLRW model used in the literature, for the other two forms, see Lecture notes.

2. Riemann and Einstein Tensor

- Riemann tensor and Einstein tensor in a manifold of any dimension are given by

$$\begin{aligned} R^i_{\ jkm} &= \partial_k \Gamma^i_{\ mj} - \partial_m \Gamma^i_{\ kj} + \Gamma^i_{\ ks}\Gamma^s_{\ mj} - \Gamma^i_{\ ms}\Gamma^s_{\ kj} && (26) \\ R_{ik} &= R^m_{\ imk} && (27) \\ R &= R^i_{\ i} && (28) \\ G_{ik} &= R_{ik} - \frac{1}{2} R\, g_{ik} && (29) \\ \nabla_m G^m_{\ i} &= 0 . && (30) \end{aligned}$$

- We can add a Λ–term to the Einstein tensor (Einstein 1917)

$$G_{ik} + \Lambda g_{ik} = \frac{8\pi G}{c^4}\, T_{ik} , \tag{31}$$

since the metric is covariant constant, i.e. $\nabla_m g_{ik} = 0$. This term also occurs from vacuum energy density ρ_V with pressure $P = -\rho_V$, and then taken to the left hand side of Einstein's equations. For this reason, this term is now called **Dark Energy** (DE).
In this case, the EoS for DE is given by $w = -1$, where $P = w\,\rho$.

- For the **Planck Units** $\mapsto$ Lecture Notes.
- For the **Limits of Einstein's theory** $\mapsto$ Lecture Notes.

3. The FLRW–Cosmos

- For the **Friedmann equations** $\mapsto$ Lecture Notes.
- For the **density parameters** $\mapsto$ Lecture Notes.
- For the **density evolution** $\mapsto$ Lecture Notes.
- The **density of relativistic particles**

$$\rho_r = \rho_\gamma + 3\rho_\nu = \rho_\gamma \left(1 + 3 \times \frac{7}{8} \times \left[\frac{4}{11}\right]^{4/3}\right) = 3.363\, a_{SB} T^4/2\,. \tag{32}$$

  $\rho_\gamma = a_{SB} T^4$, where $a_{SB} = 7.567 \times 10^{-16}$ Joule m^{-3} K^{-4} is the Stefan-Boltzmann radiation constant.
  The neutrino temperature is lower than the photon temperature due to heating by electron-positron annihilation! $T/T_\nu = 1.401$.

- The present **matter density** in the Universe follows from matter in galaxies and clusters of galaxies (kinematics, X-ray gas and gravitational lenses).
- The **deSitter** solution is a solution of the Friedmann equation for pure vacuum energy $\mapsto$ Lecture Notes.
  The luminosity distance is

$$d_L(z) = rR_0(1+z) = \frac{c}{H_0}\, z(1+z)\,. \tag{33}$$

- The **Einstein-deSitter** solution is a solution of the Friedmann equation without vacuum energy and for flat spaces, $k = 0$, $R(t) = R_0\,(t/t_0)^{2/3}$.
  The luminosity distance is

$$d_L(z) = rR_0(1+z) = \frac{2c}{H_0}\,[1 + z - \sqrt{1+z}]\,. \tag{34}$$

  Since $\sqrt{1+z} \simeq 1 + z/2$ for $z \ll 1$, this provides the Hubble-law for low redshifts.

- **FLRW Klein–Gordon equation** for $\Phi(t, x^i)$, using $\Gamma^\mu_{\mu\rho} = \partial_\rho\sqrt{-g}/\sqrt{-g}$,

$$\boxed{\Box\Phi \equiv \nabla_\mu \nabla^\mu \Phi = \frac{1}{\sqrt{-g}}\, \partial_\mu \left(\sqrt{-g}\, g^{\mu\sigma} \partial_\sigma \Phi\right) = -\left[\ddot{\Phi} + 3H\dot{\Phi} - \frac{1}{R^2}\, \nabla^2 \Phi\right]}\,. \tag{35}$$